STABILITY OF THE LONGITUDINAL BUNCHED-BEAM COHERENT DIPOLE MODE

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Abstract

The evolution of the coherent synchrotron frequency for the dipole mode with respect to the incoherent band is discussed analytically both in the unstable and stable regions. In the unstable region, Besnier's picture is recovered in the case of a capacitive impedance below transition or inductive impedance above transition. A general plot gathering all the results in both the unstable and stable regions is given. Finally, Sacherer's stability criterion is extended to include the potential-well distortion. This result is then applied to the case of the LHC at top energy.

INTRODUCTION

Consider the case of a capacitive impedance below transition or inductive impedance above transition. It is often said [1] that the coherent synchrotron frequency remains the same as the unperturbed small-amplitude synchrotron frequency when intensity increases (coherent and incoherent effects subtract). As the incoherent frequency spread is moving downwards the following question is raised: how can the beam be stable, as it seems to be impossible, even for a very large frequency spread?

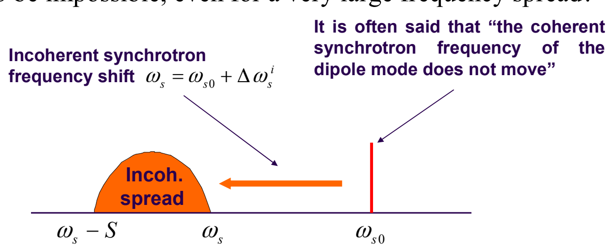

Figure 1: The case of a capacitive impedance below transition or inductive impedance above transition is considered here. How can the beam be stable?

An answer to this question was given more than twenty years ago by Besnier [2] for a parabolic distribution function. After reviewing some general results on the longitudinal bunched-beam coherent modes, another answer to this question is given in Section 2.1 for the "elliptical" spectrum, which leads to a circular range of stability. The general plot gathering all the results in both the unstable and stable regions is shown in Fig. 3. Note that Sacherer's stability criterion [3] is recovered for the stability limit. Note also that it is the same stability criterion as the one used in Ref. [4] and derived in Ref. [5] (with the approximation $3/\pi \approx 1$).

Finally, the stability criterion, taking into account the potential-well distortion, is derived analytically in Section 2.2, and applied to the case of the LHC at top energy in Section 3.

THEORY

General plot for the coherent synchrotron frequency vs. the incoherent band

The stability of the longitudinal bunched-beam coherent mode $m = ..., -1, 0, 1, ...$ can de discussed from the general dispersion relation [3]

$$I_m^{-1}(\omega) = \Delta\omega^l_{cmm} \quad . \tag{1}$$

Here, $I_m(\omega)$ is the dispersion integral given by

$$I_m(\omega) = \frac{\displaystyle\int_0^\infty \frac{\hat{\tau}^{2m}}{\omega - m\,\omega_s(\hat{\tau})} \frac{dg_0(\hat{\tau})}{d\hat{\tau}} d\hat{\tau}}{\displaystyle\int_0^\infty \hat{\tau}^{2m} \frac{dg_0(\hat{\tau})}{d\hat{\tau}} d\hat{\tau}} , \tag{2}$$

and $\Delta\omega^l_{cmm} = \omega_{cmm} - m\,\omega_{s0}$ is the coherent synchrotron frequency shift given by Sacherer's formula [6,7]

$$\Delta\omega^l_{cmm} = \frac{|m|}{|m|+1} \times \frac{j\,I_b\,\omega_s}{3\,B^3 \hat{V}_T\,h \cos\phi_s} \times \left[\frac{Z_l(p)}{p}\right]^{eff}_{mm} , \tag{3}$$

with

$$\left[\frac{Z_l(p)}{p}\right]^{eff}_{mm} = \frac{\displaystyle\sum_{p=-\infty}^{p=+\infty} \frac{Z_l(\omega^l_p)}{p} h_{mm}(\omega^l_p)}{\displaystyle\sum_{p=-\infty}^{p=+\infty} h_{mm}(\omega^l_p)} , \tag{4}$$

$$h_{mm}(\omega) = \frac{\tau_b^2}{2\pi^4}(|m|+1)^2 \frac{1 + (-1)^{|m|}\cos[\omega\,\tau_b]}{\left\{[\,\omega\,\tau_b/\pi]^2 - (|m|+1)^2\right\}^2} . \tag{5}$$

Here, $g_0(\hat{\tau})$ is the stationary distribution of the synchrotron oscillation amplitude $\hat{\tau}$, $\omega_s = 2\pi f_s$ is the synchrotron angular frequency taking into account the Potential-Well Distortion PWD (the unperturbed synchrotron angular frequency is $\omega_{s0} = 2\pi f_{s0}$), $j = \sqrt{-1}$ is the imaginary unit, $I_b = N_b\,e\,f_0$ is the current in one bunch with N_b the number of protons in the bunch, e the elementary charge, and $f_0 = \Omega_0/2\pi$ the revolution frequency, $B = \tau_b\,f_0$ is the bunching factor with τ_b the total bunch length (in seconds) taking into account the

PWD (the unperturbed total bunch length is τ_{b0}), $\hat{V}_T$ is the total (effective) peak voltage taking into account the PWD (the peak RF voltage is $\hat{V}_{RF}$), h is the harmonic number, ϕ_s is the RF phase of the synchronous particle ($\cos\phi_s > 0$ below transition and $\cos\phi_s < 0$ above) taking into account the PWD (the unperturbed synchronous phase is ϕ_{s0}), Z_l is the longitudinal coupling impedance, $\omega_p^l = p\,\Omega_0 + m\,\omega_s$ with $-\infty \le p \le +\infty$, and h_{mm} describes the bunch spectrum (sinusoidal modes for parabolic bunches).

The stability diagram for the smooth distribution function $g_0(\hat{\tau}) \propto (1-\hat{\tau}^2)^2$ used by Sacherer [3] is represented in Fig. 2, as well as the one corresponding to his "approximate" stability criterion $S \ge 4\,|\Delta\omega_{cmm}^l| / \sqrt{|m|}$ (following the example of Keil and Schnell for coasting beams [8], Sacherer approximated the stability boundaries by semi-circles). The case of a capacitive impedance below transition or inductive impedance above transition corresponds to $\mathrm{Re}\,(\Delta\omega_{cmm}^l / S) > 0$ and $\Delta\omega_S^i < 0$. Here S is the full spread between the centre and the edge of the bunch. A good approximation of the frequency spread is given by [9,10]

$$S = \left(1 + \frac{5}{3}\tan^2\phi_s\right)\frac{\pi^2}{16}\left(h\,B\right)^2\omega_s\,. \qquad (6)$$

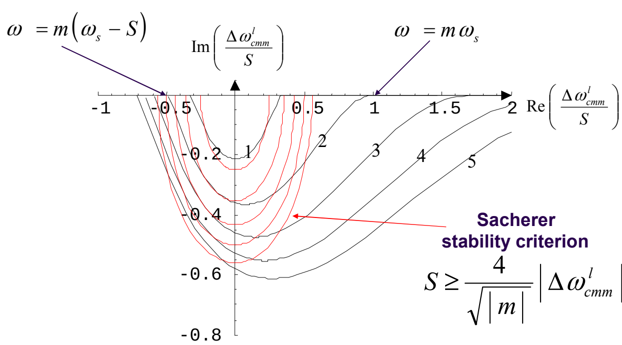

Figure 2: Stability diagrams for the smooth distribution function $g_0(\hat{\tau}) \propto (1-\hat{\tau}^2)^2$ used by Sacherer [3], approximated by semi-circles, for modes m from 1 to 5.

In the case of an "elliptical spectrum" $\hat{\tau}^2\,dg_0(\hat{\tau}^2)/d\hat{\tau}^2 \propto \sqrt{1-(2\hat{\tau}^2-1)^2}$ [11], the dispersion relation writes [10]

$$\omega_{c11} - \left(\omega_s - \frac{S}{2}\right) - 2U - j\left[\sqrt{\left(\frac{S}{2}\right)^2 - \left[\omega_{c11} - \left(\omega_s - \frac{S}{2}\right)\right]^2} - 2V\right] = 0. \qquad (7)$$

Here, the coherent synchrotron frequency shift has been written $\Delta\omega_{c11}^l = U - j\,V$. Motions $\propto e^{j\omega t}$ are considered, which means that the beam is unstable when $V > 0$. Furthermore, the usual case where the resistive part of the impedance is small compared to the imaginary part, is assumed, i.e. $V \ll |U|$. Following Besnier's approach [2], Fig. 3 is obtained.

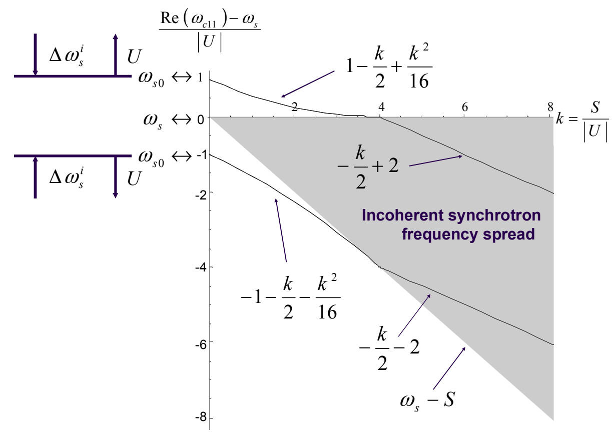

Figure 3: Evolution of the coherent synchrotron frequency for the dipole mode with respect to the incoherent frequency spread, for an elliptical spectrum.

This plot is very similar to Besnier's graph (see Fig. 4 of Ref. [2]), except that, contrary to Besnier, Fig. 3 also predicts stability for the case of an inductive impedance below transition or a capacitive impedance above transition. This difference comes from the fact that Besnier used a parabolic distribution function, which introduces some pathologies in the stability diagrams, due to its sharp edge [10]. It is seen in Fig. 3 that in the absence of frequency spread ($S = 0$ and thus $k = 0$), the coherent synchrotron frequency ω_{c11} is close to the unperturbed small-amplitude synchrotron frequency ω_{s0} [10]. When the synchrotron frequency spread increases, the coherent synchrotron frequency ω_{c11} moves closer and closer to the incoherent band (stable region). The two possible cases are represented in Fig. 3: the case of a capacitive impedance below transition or inductive impedance above transition corresponds to $U > 0$ and $\Delta\omega_S^i < 0$ (and thus $\omega_s < \omega_{s0}$), and the case of a capacitive impedance above transition or inductive impedance below transition corresponds to $U < 0$ and $\Delta\omega_S^i > 0$ (and thus $\omega_s > \omega_{s0}$). Beam stability is obtained when the coherent synchrotron frequency ω_{c11} enters into the incoherent band, i.e. when $\omega_{c11} = \omega_s$ for the case of a capacitive impedance below transition or inductive impedance above transition, and when $\omega_{c11} = \omega_s - S$ for the case of a capacitive impedance above transition or inductive impedance below transition. In both cases, the stability limit is reached for $k = 4$, i.e.

$S = 4|U|$, which is Sacherer's stability criterion (in the usual approximation $V << |U|$).

Starting with a low intensity beam, the coherent synchrotron frequency ω_{c11} lies in the middle of the incoherent band, $\omega_{c11} = \omega_{s0} - S/2$. As the intensity increases, the coherent synchrotron frequency ω_{c11} moves closer and closer to the limit of the incoherent band. Beam stability is lost when the coherent synchrotron frequency ω_{c11} moves out of the incoherent band, i.e. when $\omega_{c11} = \omega_s$ for the case of a capacitive impedance below transition or inductive impedance above transition, and when $\omega_{c11} = \omega_s - S$ for the case of a capacitive impedance above transition or inductive impedance below transition. Again, in both cases, the stability limit is reached for $k = 4$, i.e. $S = 4|U|$.

Stability Criterion Taking into Account the Potential-Well Distortion

The stability limit obtained above is the same as Sacherer's stability criterion. Using Eq. (3), with $\hat{V}_T(I_b)$ and $B(I_b)$, but neglecting the synchronous phase shift (which comes from the resistive part of the impedance, usually small), the following stability criterion is obtained [10]

$$I_b \le \left(1 + \frac{5}{3}\tan^2\phi_{s0}\right)\frac{3\pi^2}{32} \times \frac{h^3\,\hat{V}_{RF}\,|\cos\phi_{s0}|\,B_0^5}{\left|\frac{Z_l(p)}{p}\right|_{11}^{eff}} \times F_{PWD}\,, \quad (8)$$

with

$$F_{PWD} = \sqrt{\frac{1}{2}\left(a + \sqrt{a^2+4}\right)}\,, \quad (9)$$

$$a = \frac{9}{32}\left(1 + \frac{5}{3}\tan^2\phi_{s0}\right) h^2\,B_0^2\,Sgn(\cos\phi_{s0})\,\frac{j\left[\frac{Z_l(p)}{p}\right]_{00}^{eff}}{\left|\frac{Z_l(p)}{p}\right|_{11}^{eff}}\,. \quad (10)$$

APPLICATION TO THE LHC AT TOP ENERGY

The most dangerous longitudinal single-bunch effect in the LHC is the possible suppression of Landau damping at top energy (7 TeV) [12]. The stability criterion of Eq. (8) is the same as the one used in Ref. [13], on the flat top and neglecting the potential-well distortion ($F_{PWD} = 1$). Using the same numerical values, $|Z_l(p)/p|_{00}^{eff} = |Z_l(p)/p|_{11}^{eff} = 0.28\,\Omega$, $E = 7$ TeV, $\sigma_b = 7.5$ cm, $\hat{V}_{RF} = 16$ MV, and $h = 35640$, the same intensity threshold is obtained (since $F_{PWD} \approx 1.01$)

$$N_b^{th} = 2.4 \times 10^{11}\ \text{p/b}\,. \quad (11)$$

CONCLUSION

The evolution of the coherent synchrotron frequency for the dipole mode has been described analytically both in the unstable and stable regions. The general plot gathering all the results in both the unstable and stable regions is represented in Fig. 3. The stability criterion, taking into account the potential-well distortion, has also been derived analytically and applied to the case of the LHC at top energy. The same numerical result as in Ref. [13] has been obtained as the potential-well distortion is very small in this case.

ACKNOWLEDGEMENTS

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