Scaling laws for crossing of space charge resonances *

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Abstract

Crossing of intrinsic space charge resonances may lead to emittance variations depending on the strength of space charge, the crossing rate and the lattice. We present scaling laws for the the “Montague-” coupling resonance near $2Q_x - 2Q_y = 0$, and for the fourth order structure resonance near $4Q_y = 12$, which are expressed in terms of simple algebraic expressions and benchmarked on fully self-consistent particle-in-cell simulations in the 2D coasting beam limit.

INTRODUCTION

In a recent study on the Montague-resonance emittance coupling it was shown that relatively simple and universal scaling expressions can be derived by comparing analytical results and self-consistent particle-in-cell simulations [1]. The Montague-resonance is an example of a space charge (difference) resonance [2], which is driven by the zeroth harmonic of the lattice functions, e.g. $2Q_x - 2Q_y \approx 0$. It is therefore expected to be “immune” to the actual lattice beta function, which is also confirmed by simulation [1]. For the fourth order structure resonance, $4Q_y \approx 12$, one expects that the weight of the underlying Fourier harmonic matters, and that the smoother focusing of a triplet cell shows less growth than a doublet cell as predicted by expanding the analytical potential in Ref. [3].

MONTAGUE RESONANCE

In Fig. 1 we first show an example of stop-band using the final rms emittances, and by varying $Q_{0,x}$ in small steps. Results are obtained with the MICROMAP-library [4] employing 50.000 particles and a 128x128 grid with conducting boundary conditions on a square box of width 6 times the horizontal rms size of the beam. We present an example for parameters borrowed from measurements at the CERN Proton Synchrotron [5, 1]: a fixed vertical working point $Q_{0,y} = 6.21$ and an emittance ratio of $\epsilon_x/\epsilon_y = 3$ are used, while the absolute values of initial normalized rms emittances are chosen as $\epsilon_x = 2.5\pi$ mm-mrad and $\epsilon_y = 7.5\pi$ mm-mrad. The current is set to yield a maximum vertical tune shift of $\Delta Q_y = -0.105$ in the center of a Gaussian distribution, which leads to a maximum horizontal tune shift of $\Delta Q_x = -0.061$ for the given emittance ratio. The plotted values, where each marker is a simulation with different $Q_{0,x}$, are defined as averages of the rms emittance values between turn 1000 and 2000, which gives a good measure of the saturation stage. In order to justify

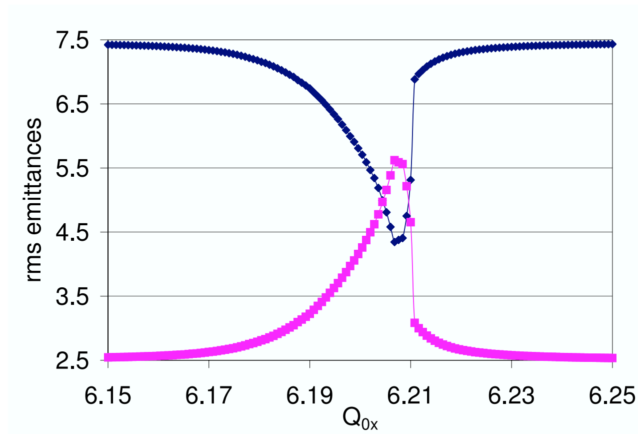

Figure 1: Static tunes stop-band: final rms emittances for different values of $Q_{0,x}$.

the use of constant focusing for the present study, we have compared it with (linear) periodic focusing and find that the difference is negligible. The width of the stop-band (in terms of the horizontal tune width Θ) was found to be [1]:

$$\Theta = \frac{3}{2}(\sqrt{\epsilon_r} - 1)\Delta Q_x, \tag{1}$$

where ΔQ_x is the incoherent tune spread (in KV-equivalence, hence half of the maximum tune spread of a Gaussian beam), and ϵ_r is the ratio of initial transverse emittances (here assumed ≥ 1, without loss of generality). Similarly the number of betatron periods for emittance exchange N_{ex} was found to scale approximately as

$$N_{ex}^{-1} \approx \frac{\Delta Q_x}{Q_{0,x}}. \tag{2}$$

In order to derive a scaling law for the dynamical crossing, we proceed in the following way: using the linear dependence on ΔQ_x in both, Eq. 1 and Eq. 2 under static conditions, we postulate that the exchange for fast crossing depends quadratically on ΔQ_x. In fact, we have found that the emittance growth is unchanged, if the actual width of the stop-band - $\propto \Delta Q_x$ - is crossed during a time, which is inversely proportional to ΔQ_x. Hence, the emittance

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growth depends on the quantity

$$\frac{(\Delta Q_x)^2}{\dot{Q}}, \tag{3}$$

where $\dot{Q}$ is defined as tune change per turn. As an example for dynamical crossing we use the standard case of Fig. 1 and move the working point $Q_{0,x}$ starting from the side of lower tunes over the range $6.15 \leq Q_{0,x} \leq 6.27$ enclosing the stop-band. For this crossing "from below"we apply a linear tune ramp in time. In Fig. 2 we show the evolution of emittances as function of the instantaneous tune for two cases, where the crossing of the same tune range is performed in 100, respectively 1000 turns. It is noted that for the 100 turns case the final emittances are practically equal; for the 1000 turns case the final emittances are basically exchanged with the initial emittances. The complete picture

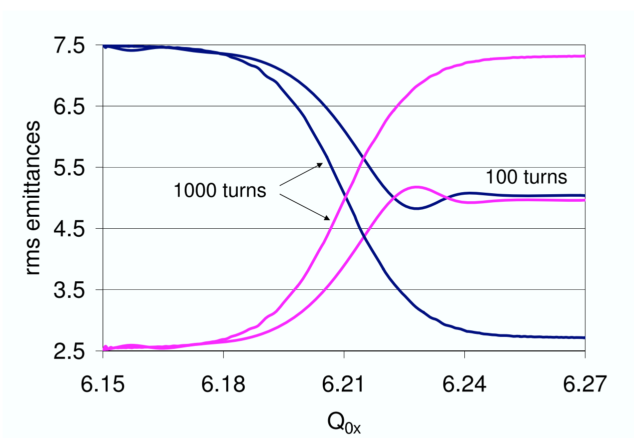

Figure 2: Evolution of emittances by crossing the stop-band dynamically from below over 100 and 1000 turns.

of the final emittances after crossing the band at variable number of turns is shown in Fig. 3. In the 100 turns case

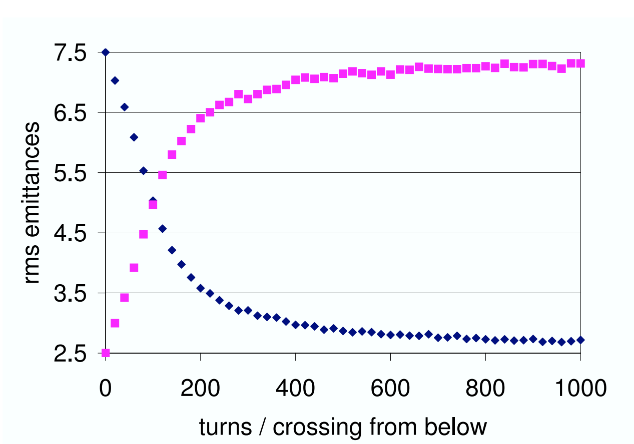

Figure 3: Final emittances after crossing $Q_{0,x} = 6.15 \rightarrow 6.27$ at variable rates.

the essential part of the stop-band in Fig. 1, which has a tune width of 0.04, is crossed in 33 turns or 205 betatron periods. This time agrees with the fastest rise time of the static case, which therefore sets the time-scale needed for a crossing to just equalize final emittances.

Note that for faster crossing the exchange is only partial, with a linear dependence on the number of turns. Hence we find that with $N < N_{ex}$ the emittance exchange is proportional to the inverse tune changing rate. Using Eq. 3 and Eq. 1 for the stop-band width we therefore suggest a scaling law for the relative growth of the smaller emittance (here in y)

$$\frac{\Delta\epsilon}{\epsilon} = \alpha_M \frac{(\sqrt{\epsilon}_r - 1)^2 \Delta Q_x^2}{\dot{Q}}, \tag{4}$$

assuming that the full stop-band is crossed at constant rate and $\dot{Q}$ is the change of the horizontal tune per turn. α_M is a factor, which we determine from the graph of Fig. 3 as $\alpha_M \approx 1.8$.

Crossing in the opposite direction leads to a significantly suppressed emittance coupling due to the fact that the space charge de-tuning points also downwards in the tune diagram (details see Ref. [1]).

FOURTH ORDER STRUCTURE RESONANCE.

This mode is another purely space charge driven resonance, which was first studied by simulation in connection with heavy ion inertial fusion [6]. There it was suggested that this mode should be observable in a ring lattice, where the phase advance per cell of 90^0 is approached from above by means of increasing space charge during a bunch compression. It was also found that the fourth order structure resonance can be in a competition with the envelope instability in a ring, which would normally be observable at a slightly higher current level [8]. More recently, it was observed experimentally and confirmed by simulation in a study of foil injection into the KEK synchrotron [7].

In Fig. 4 we show a phase space plot obtained for a simulation of the SIS18 lattice, which has 12 super-periods with the option of triplet or doublet focusing. The vertical phase advance per super-period is close to 100^0, hence $Q_{0,y} \approx 3.3$ in normal operation. For our example we have chosen a tune ramp crossing symmetrically the value $Q_{0,y} = 3$. The scatter plot shows the typical result of particles trapped in fourth order resonance islands, which move to larger amplitudes to compensate the lowered bare tune by weaker space charge.

In our attempt to find a scaling law we first study the dependence on the number of turns Δn, here defined as number of turns to change the tune by ΔQ_x (in this section understood as full space charge tune spread of a Gaussian distribution), for crossing at different levels of space charge. In contrast with the Montague case we find a quadratic dependence on the turn number for faster crossing, and a linear regime for slow crossing, with significant growth. This

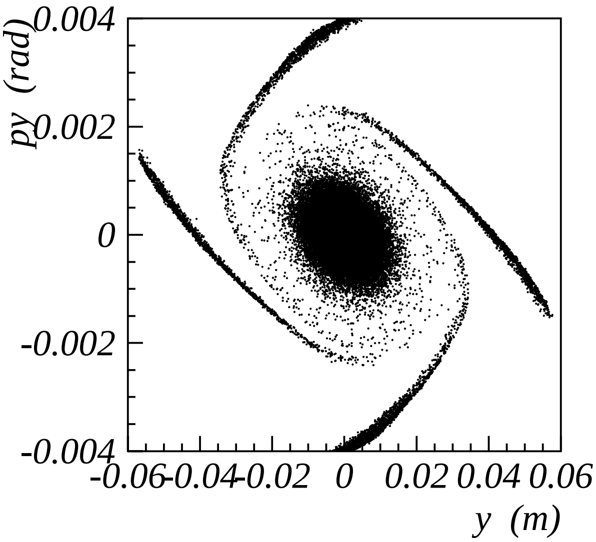

Figure 4: Vertical phase space plot after crossing the resonance condition $Q_{0,y} = 3$.

is interpreted as a result of trapping of particles in the resonance islands, which move to larger amplitudes due to the combined effect of tune lowering and space charge noted in Fig. 4. Note that for slower crossing an increasing number of particles is involved in the trapping. We find that curves for different values of ΔQ_x can be brought to overlap with good accuracy, if Δn is multiplied by a suitably chosen scaling factor α as shown in Fig. 5.

As we are mostly interested in the onset region of emittance growth, we focus here on the quadratic regime, which extends approximately up to a doubling of rms emittances. Surprisingly, we find that α can be fitted quite well with a linear expression

$$\alpha \approx \alpha_0 \Delta Q_x. \tag{5}$$

We find that α_0 depends on the specific lattice properties. For constant focusing, obviously, $\alpha_0 = 0$. For our example of a relatively smooth triplet focusing we obtain $\alpha_0 \approx 9$. For a restricted choice of parameters we have also tested a doublet focusing with otherwise identical parameters and found that α_0 roughly doubles.

The resulting scaling law in terms of a space charge independent tune rate is

$$\frac{\Delta\epsilon}{\epsilon} \approx \alpha_4 \frac{\Delta Q_x^4}{\dot{Q}^2}, \tag{6}$$

where α_0^2 has been absorbed into the fore-factor, for which triplet focusing yields $\alpha_4 \approx 4 \cdot 10^{-4}$, and doublet focusing about four times this value. The behavior of much stronger response for doublet focusing has been found in Ref. [3], where a g-factor was introduced as measure for the strength of the Fourier component in the space charge potential, which drives the resonance.

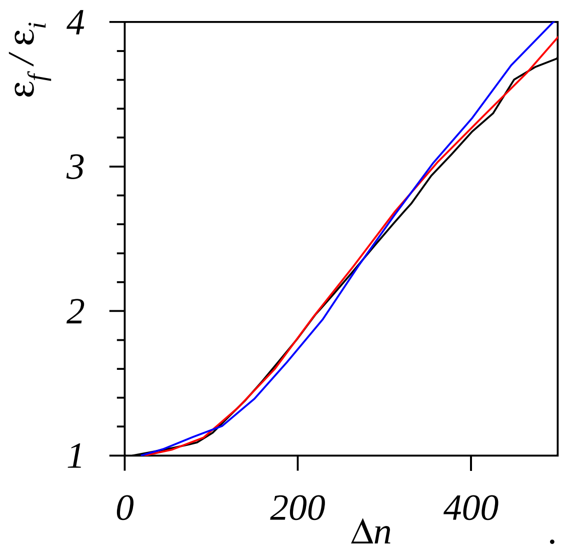

Figure 5: Final rms emittance growth factors as function of Δn and for different space charge tune shifts (0.2/black, 0.4/red, and 0.8/blue

SUMMARY AND CONCLUSION

Our scaling relationships show that only few parameters are needed to predict the effect of resonance crossing on rms emittances. In the Montague case the scaling depends on $\frac{\Delta Q_x^2}{\dot{Q}}$, and in the fourth order structure resonance case in the weak emittance growth regime on $\frac{\Delta Q_x^4}{\dot{Q}^2}$. In the latter case the growth is enhanced for a doublet focusing due to less smooth beta functions.

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