FOCUSING IN RF ACCELERATING GAPS WITH ASYMMETRICALLY CURVED ELECTRIC EQUIPOTENTIALS

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Abstract

An analysis is made of the extra focusing that can be obtained by suitably shaping the dee gap, in addition to the usual 'phase change', 'velocity gain' and strong focusing effects. To obtain this extra focusing the electric equipotentials must be curved in the median plane; the extra focusing power is shown to be proportional to the integral $\int(d^2y/dx^2)dV$ across the gap of the second derivative of the equipotential curves $y(x,V)$ in the median plane. There is also a complementary relationship between the radial and vertical focal powers of the dee gap; their sum is shown to be proportional to the rate of change of energy gain with phase, provided space charge and transit time effects can be neglected. The application of these theorems to central region design is illustrated by a number of examples.

1. Introduction

The electrical focusing properties of dee gaps whose shape is uniform along the gap have been described in classic studies by Rose and others.[1] Near the point at which ions are injected into a cyclotron, however, the dee shape is generally quite non-uniform and strong additional focusing effects can arise associated with curvature of the median plane equipotentials. These effects must be properly taken into account if accurate estimates of the phase acceptance and beam behaviour near the centre are to be obtained. In this paper we begin by considering the interrelation between the focal properties and the energy gain at the dee gap, and their dependence on the gap shape and the RF phase. By suitably shaping the field with a non-uniform dee gap (as in Fig. 1) it is shown how a useful change in ν_z^2 can be obtained in a region where the vertical focusing is generally weak, and how this is accompanied by an opposing change in ν_r^2 which is equal and opposite for the phase of maximum energy gain. Finally, a simple expression is derived for estimating the magnitude of these changes in focusing directly from the curvature of the median plane equipotentials. This method has proved very helpful in selecting possible electrode shapes for the TRIUMF central region, and is illustrated by examples taken from these studies.

2. Dee Gap Focusing-Energy Gain Relationship

The electric field strength in the dee gap may be written $\vec{E}\cos\omega t$ where ω represents the RF angular frequency. An ion of charge q crossing the gap will experience a momentum change

$$\vec{\Delta p} = q\int_{-\infty}^{+\infty}\vec{E}(x,y,z)\cos\omega t\,dt \tag{1}$$

where the origin will be assumed to lie on the dee gap centre line, x being measured outward along it, y normal to it and z vertical. Now, neglecting 'velocity gain' effects,[1] y and t are related by

$$\omega t = \phi + \omega y/v = \phi + hy/R \tag{2}$$

where ϕ is the RF phase when the ion is at y=0, R is the equilibrium orbit radius for an ion of velocity v, and h is the acceleration harmonic (h=5 for TRIUMF). Thus for the vertical focal length f_z we have, making a first-order Taylor expansion of E_z and changing to y as the independent variable

$$\frac{1}{f_z} = -\frac{q}{mv^2}\int_{-\infty}^{+\infty}\frac{\partial E_z}{\partial z}\cos\left(\frac{hy}{R}+\phi\right)dy. \tag{3z}$$

A similar equation (3x) holds for $1/f_x$, with z everywhere replaced by x. In the absence of space charge div $\vec{E}$ = 0, so that we find

$$\frac{1}{f_x}+\frac{1}{f_z} = \frac{q}{mv^2}\left\{\left[E_y\cos\left(\frac{hy}{R}+\phi\right)\right]_{-\infty}^{\infty} + \frac{h}{R}\int_{-\infty}^{+\infty}E_y\sin\left(\frac{hy}{R}+\phi\right)dy\right\}$$

where we have integrated by parts. Now $E_y \to 0$ as $y \to \pm\infty$

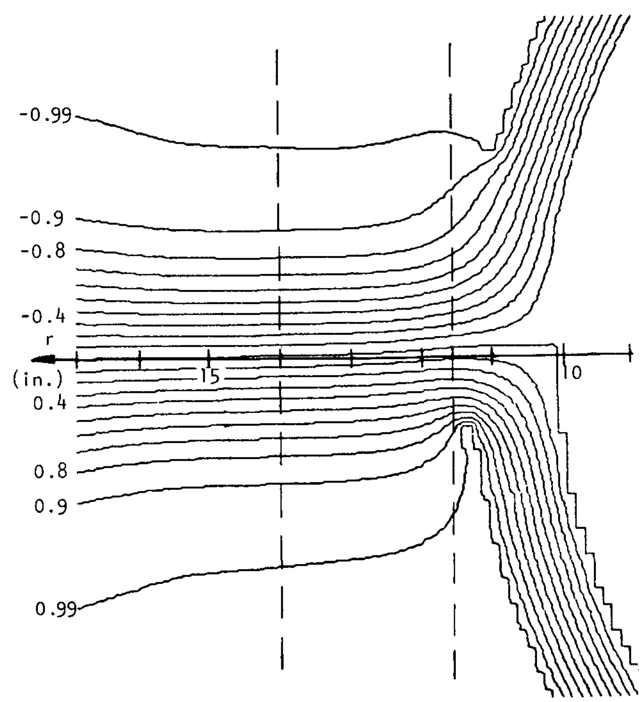

Fig. 1. Median plane equipotentials shaped for improved focusing on the first turn (r ≈ 12 in.) in TRIUMF.

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and therefore

$$\frac{1}{f_x} + \frac{1}{f_z} = -\frac{h}{2RT_c}\frac{\partial}{\partial\phi}(\Delta T) \tag{4}$$

—the sum of the x- and z-focal powers should be proportional to the rate of change with phase of ΔT, the energy gained by the ion at the gap,

$$\Delta T = q\int_{-\infty}^{+\infty} E_y \cos\left(\frac{hy}{R}+\phi\right) dy. \tag{5}$$

The velocity v is assumed constant across the gap in the derivation, but for practical purposes we identify $\frac{1}{2}mv^2$ with T_c, the ion energy at the gap centre. Relation (4) can be made dimensionless by writing the focal powers in terms of their contributions $\Delta\nu^2=R/\pi f$ to their respective betatron oscillation frequencies (for f>>R). Thus

$$\Delta\nu_r^2 + \Delta\nu_z^2 = -\frac{h}{2\pi T_c}\frac{\partial}{\partial\phi}(\Delta T). \tag{6}$$

Eq.(4) is clearly a generalized form of the familiar $f_x^{-1}+f_z^{-1}=0$ applying to non-accelerating electric and magnetic lenses. This exactly complementary relation also applies to our dee gap lenses for the prime phase ϕ_m for which the energy gain is a maximum: the vertical and horizontal focal powers are there equal and opposite. Thus additional vertical focusing $\Delta\nu_z^2$ can be obtained, at the expense of allowing ν_r^2 to be reduced by the same amount. This can be advantageous, as was found for TRIUMF,[2] where a lower ν_r helps to reduce the effects of radial longitudinal coupling due to the $\nu_r=1$ resonance.

3. Dependence on Phase and Dee Gap Asymmetry

The sinusoidal dependence of the focal power $1/f_z$ on phase and the effect of the y-symmetry of the gap may be brought out by writing (3z) in the form

$$\frac{1}{f_z} \equiv F_z \sin(\phi-\phi_z) \tag{7z}$$

$$\equiv F_z^S \sin\phi - F_z^A \cos\phi. \tag{8z}$$

Here ϕ_z is defined as the transition phase between vertical defocusing ($\phi<\phi_z$) and focusing ($\phi>\phi_z$). The 'symmetric' and 'asymmetric' focal powers are given explicitly by

$$F_z^{\substack{S\\A}} = F_z \begin{matrix}\cos\\ \sin\end{matrix}\phi_z \equiv \frac{q}{mv^2}\int_{-\infty}^{\infty}\frac{\partial E_z}{\partial z}\begin{matrix}\sin\\ \cos\end{matrix}\left(\frac{hy}{R}\right)dy. \tag{9z}$$

The appropriateness of this terminology can be seen by considering a dee gap symmetric about the plane y=0; in this case E_z is an odd function of y so that $F_z^A=0$, $\phi_z=0$ and $1/f_z=F_z^S\sin\phi$.

The phase dependence of $1/f_x$ may be treated in the same way. Thus we are led to introduce quantities F_x, ϕ_x, F_x^S and F_x^A defined in new equations (7x), (8x) and (9x) exactly analogous to Eq.(7z), etc. above. For a y-symmetric dee gap, $F_x^A=0$, $\phi_x=0$ and $1/f_x=F_x^S\sin\phi$.

For the energy gain we may write (5) as

$$\Delta T \equiv \Delta T_m \cos(\phi-\phi_m) \tag{10}$$

$$\equiv \Delta T_m^S \cos\phi + \Delta T_m^A \sin\phi \tag{11}$$

where this equation defines ΔT_m, ΔT_m^S and ΔT_m^A. For a y-symmetric gap, $\Delta T_m^A=0$, $\phi_m=0$ and $\Delta T=\Delta T_m^S\cos\phi$.

Substituting (7) and (10) in the energy gain-focusing relation (4) and eliminating ϕ, we find

$$F_x e^{i\phi_x} + F_z e^{i\phi_z} = \frac{h}{2RT_c}\Delta T_m e^{i\phi_m} \tag{12}$$

i.e. the focal powers $1/f_x$ and $1/f_z$ can be represented in an Argand diagram as vectors whose sum represents a certain fraction of the ΔT vector. In general, therefore, if ϕ_z is more negative than the phase of maximum energy gain, ϕ_x will be more positive, and vice-versa:

$$\phi_x \gtrless \phi_m \gtrless \phi_z. \tag{13}$$

In the special case of a uniform (parallel) dee gap, the case which Rose treats, the radial focal power $1/f_x=0$, and therefore by (12) $\phi_z=\phi_m$. Also, substituting for F_z from (12) into (7z), we find

$$\frac{1}{f_z} = \frac{h\Delta T_m}{2RT_c}\sin(\phi-\phi_m) \tag{14}$$

which is just Rose's formula for the 'phase change' focusing term (he takes ϕ_m as his zero phase). Now, as Rose has pointed out, if there is uniformity along the dee gap, the degree of asymmetry across the gap cannot influence its effective focal power because $\phi_z=\phi_m$, i.e. the vertical transition phase is always the same as the prime phase of maximum energy gain (in the approximation that velocity gain effects may be neglected). For a non-uniform dee gap the same is true ($\phi_z=0°=\phi_m$) if the gap is symmetric across y=0. To obtain improved vertical focusing around the prime phase ϕ_m, the gap must be both non-uniform and asymmetric, giving curved asymmetric equipotentials in the median plane (Fig. 1) as well as in the vertical plane (Fig. 2).

Han and Reiser's results[3] on the focal properties of non-uniform dee gaps have been used to check the validity of some of the above relations. They tracked ion paths by numerical integration through dee gaps of various designs for many energies and phases, parametrizing their results in terms of lens parameters. Figs. 3 and 4 show two examples of their data (the points) for the variation of focal

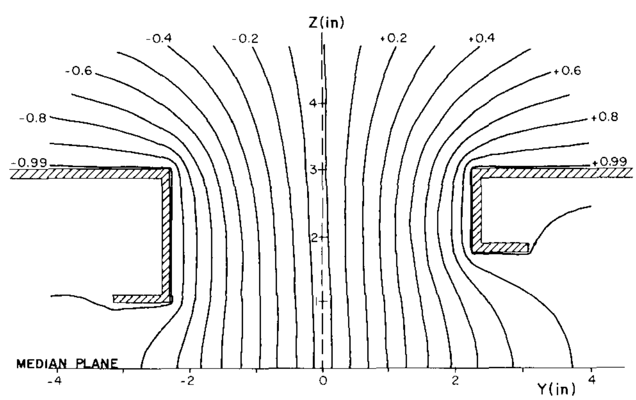

Fig. 2. Equipotentials in a vertical plane across an asymmetric dee gap.

power and $\partial\Delta T/\partial\phi$ with phase for T_c=0.2 MeV$\approx\Delta T_m$. In Fig. 3 the gap is symmetric with a pair of vertical grid posts on either side (see inset). leaving a beam aperture 1.0 in. wide × 3.0 in. high. In Fig. 4 the gap is asymmetric with posts on only one side. These two cases have been chosen not so much to show the excellence of fit, which is much better for larger values of $T_c/\Delta T_m$, but to illustrate how well the relations hold up for $T_c/\Delta T_m \simeq 1$, where velocity gain and strong focusing effects would be expected to be quite important.

In both the figures there is a nearly sine-like variation of focal power and energy gain with phase, in agreement with (7) and (10). The curves through the points are least squares harmonic fits; two harmonics were sufficient in all cases, the amplitude of the second harmonic never being more than 15% of the first and usually much smaller. The presence of the second harmonic component is attributable to 'velocity gain' focusing effects. For the symmetric gap the first harmonic components indicated transition phases ϕ_z=11°, ϕ_x=-3° and ϕ_m=0°; theoretically all should be 0°. For the asymmetric gap ϕ_m=9° has shifted to a lagging phase, as expected, while ϕ_z=-35° and ϕ_x=55° have moved wide on either side of ϕ_m, in accordance with (13). A considerable gain in vertical focusing has been achieved at the prime acceleration phase ϕ_m, while there has been an equally large reduction in horizontal focusing, as required by (4). The sum of $1/f_x$ and $1/f_z$ is also plotted on Figs. 3 and 4 for comparison with $-(h/2RT_c)\ \partial\Delta T/\partial\phi$. In the symmetric case they have similar shapes although the sum is distorted away from the origin by 15% zero and second harmonic components in $1/f_x$. In the asymmetric case the agreement is quite good, in spite of the $1/f_x$ and $1/f_z$ curves individually having very different shapes.

4. Focal Power from Equipotential Curvature

To estimate the strength of the focusing provided by dee gap shaping the integrals in (9) must be evaluated. These can be rewritten in terms of the curvature of the equipotentials to provide expressions which can be relatively easily evaluated from an equipotential plot. Noting that in a vertical (y-z) plane the slope of an equipotential line $dy/dz=-E_z/E_y$, and since in our approximation z is assumed independent of y within the gap field, we may write

$$F_z^{\overset{S}{A}} = \frac{q}{2T_c}\frac{d}{dz}\int_{-V_0}^{V_0}\frac{dy}{dz}\,\overset{\sin}{\cos}\left(\frac{hy}{R}\right)dV \qquad (15z)$$

where we have changed to the potential V as independent variable and where $\pm V_0$ is the dee voltage. Clearly if the equipotentials are parallel there is no focusing. Similarly, in the median plane the slope $dy/dx=-E_x/E_y$ and

$$F_x^{\overset{S}{A}} = \frac{q}{2T_c}\frac{d}{dx}\int_{-V_0}^{+V_0}\frac{dy}{dx}\,\overset{\sin}{\cos}\left(\frac{hy}{R}\right)dV \qquad (15x)$$

$$= \frac{q}{2T_c}\int_{-V_0}^{V_0}\frac{d^2y}{dx^2}\,\overset{\sin}{\cos}\left(\frac{hy}{R}\right)dV \qquad (16x)$$

i.e. the focal power depends on the integral of the second derivative—loosely, the curvature—of the equipotential curves, modified by a factor to account for the RF oscillation in field strength. The transition phase ϕ_z may be derived from $\tan\phi_z=F_z^A/F_z^S$, and similarly for ϕ_x. If the transit time is relatively short so that $hy/R \ll 1$, then for the phase of

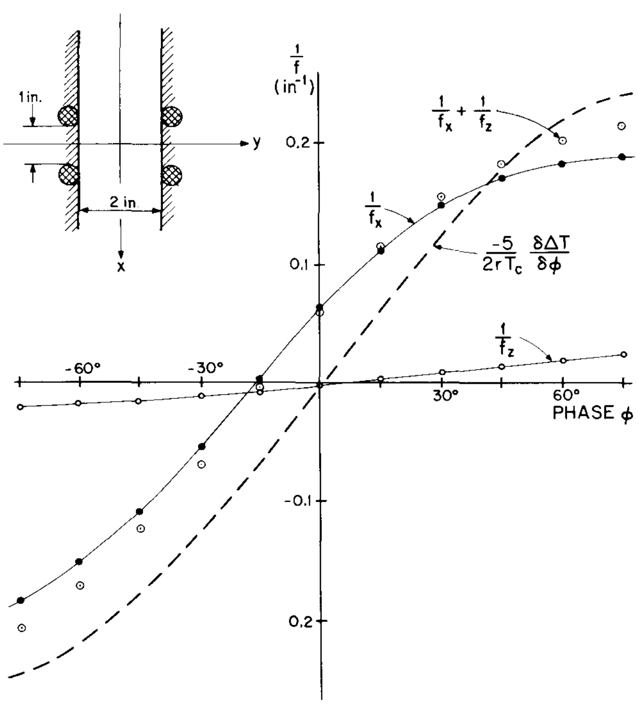

Fig. 3. Focal power vs phase for a symmetric non-uniform dee gap.

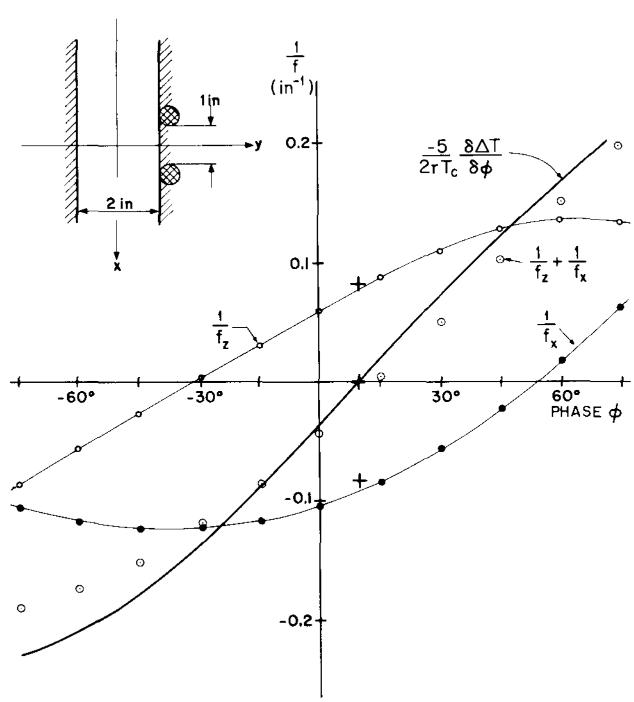

Fig. 4. Focal power vs phase for an asymmetric non-uniform dee gap.

greatest interest ϕ_m, which we assume is close to 0°,

$$\frac{1}{f_z}(\phi_m) = -\frac{1}{f_x}(\phi_m) \simeq +\frac{q}{2T_c}\int_{-V_0}^{+V_0}\frac{d^2y}{dx^2}\,dV. \qquad (17)$$

Thus we can evaluate both vertical and horizontal focal powers at the phase of maximum energy gain for all radii by examining just one equipotential plot, that for the median plane.

In practice it is easier to use formulae like (15), integrating over the slope and then finding the variation with radius, than those like (16) or (17) where the integration takes place directly over the 'curvature'. From equipotentials plotted at say 10% intervals of the dee voltage V_0, the slopes dy/dx are estimated along a trajectory, added up and multiplied by the proper constant; repeating this for trajectories at different radii the radial derivative may be obtained, giving the value of $1/f_z(\phi_m)$. As an example, Fig. 5 illustrates the focal power predicted by (17) for the field illustrated in Fig. 1, where an asymmetric non-uniform dee gap shape was introduced near the first half-turn gap crossing to provide higher vertical and lower radial focusing and hence allow better matching to the radial acceptance for positive phases;[2] (the region studied is enclosed by the two dashed lines in Fig. 1). For comparison the curve in Fig. 5 shows the results of three-dimensional numerical orbit calculations at $\phi=-4°\simeq\phi_m$ of the difference in focal power between the field shown and that for a similar non-uniform but symmetric dee gap. The agreement is quite good. For more accurate results, and for phases different from ϕ_m and 0°, one should use (15) or (16) in full, including the cos or sin (hy/R) weighting factors and computing the F^S as well as the F^A terms.

It is also possible to estimate the energy gain terms ΔT_m^S and ΔT_m^A and prime accelerating phase ϕ_m from the median plane equipotential plot. From (5) and (10), and changing to V as independent variable, we find

$$\Delta T_m^{\substack{S\\A}} = \pm q\int_{-V_0}^{V_0}\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{hy}{R}\right)dV \qquad (18)$$

and $\tan\phi_m=\Delta T_m^A/\Delta T_m^S$. For the asymmetric dee gap illustrated in Fig. 4 these formulae yield $\phi_m=9.5°$, in good agreement with the value of 9° quoted above and obtained by harmonic analysis of Han's numerical orbit tracking results.

From (17) we see that if the equipotential lines can be made parabolic so that d^2y/dx^2 is constant for each line, the focal power obtained will be uniform along the gap. In this case the focal power at ϕ_m can be calculated from the integrals of the equipotential slopes taken along just two trajectories chosen for convenience anywhere in the region:

$$\frac{1}{f_z}(\phi_m) = -\frac{1}{f_x}(\phi_m) = \frac{q/2T_c}{x_2-x_1}\int_{-V_0}^{V_0}\frac{dy}{dx}\,dV\,\Bigg|_{x_1}^{x_2}. \qquad (19)$$

A small region of this sort occurs close to any plane of mirror x-symmetry for the dee gap, e.g. midway between the vertical posts in the geometries studied by Han (Figs. 3,4) and for the 'injection gap' where the beam crosses the TRIUMF dee gap for the first time. For the case shown in Fig. 4 (19) was used to obtain $1/f_z(\phi_m)=-1/f_x(\phi_m)=0.083$. These values are plotted on Fig. 4 as upright crosses for $\phi_m=9.5°$, as determined above. There is good agreement with Han's numerical results for $1/f_z$ and $1/f_x$.

The simplicity of the formulae (15), (17), (18) and (19) makes it possible to perform rapid hand calculations of the essential focal properties of the dee gap. This can result in a considerable saving in time and money during an iterative optimization process, since the focal properties of various electrode shapes can be determined quickly and without recourse to numerical orbit tracking in three dimensions. However, such orbit calculations, based on a very accurate electric field (∿0.1%), should be performed before transforming any designs into hardware; at TRIUMF the field has been obtained by a three-dimensional relaxation programme.

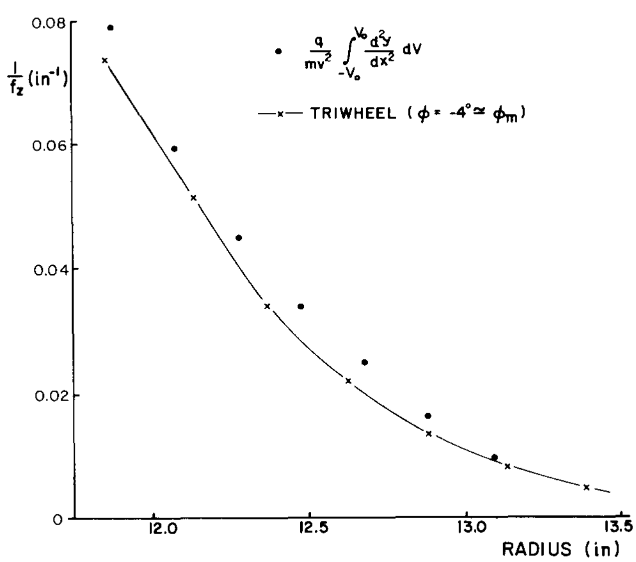

Fig. 5. Vertical focal power of the TRIUMF dee gap.

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