Proceedings of the 9th International Conference on Cyclotrons and their Applications
September 1981, Caen, France

ORBIT DYNAMICS FOR AVF CYCLOTRONS IN HARD EDGE APPROXIMATION

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Abstract. - Expressions for ν_z^2 and ν_r^2 have been derived previously by Richardson in the hard edge approximations. Taking into consideration the Thomas contribution and satisfying the isochronous requirements

$$\nu_z^2 = \frac{N}{\pi}\,\frac{B_H - B_V}{\bar{B}}\tan(k-\mu_o)$$

where k = Thomas angle, μ_o = angle of flare, B_H, B_V and $\bar{B}$ are the hill, valley and the average fields. The purpose of this paper is to show that more generalized equations may be obtained from a simplified approach based on matrix methods, wherein (a) the magnet of an AVF cyclotron is considered to be a combination of a series of dipoles having field strengths of hill and valley respectively; and (b) the spiral angles at the entrance and exit have been assumed to be unequal ($\epsilon_1 \neq \epsilon_2$). The equations of motion for this system of dipole magnets with non-normal entry and exit angles have been set up taking into account the first order effects only. It has been shown that the generalized set of equations reduced to: i) the Richardson's equations for $\epsilon_1 = \epsilon_2$, ii) the conventional equation for radial sector cyclotrons for $\epsilon_1 = \epsilon_2 = 0$ and iii) the Schatz equation for separated sector cyclotrons with $B_V = 0$.

1. Introduction. - The equations for motion of particles in an A.V.F. cyclotron have been derived in the hard edge approximation by impulse method by Richardson[1]. The expression for ν_z^2 is given by,

$$\nu_z^2 = \frac{N}{\pi}\,\frac{B_H - B_V}{\bar{B}}\tan(k-\mu_o) \qquad \ldots\ldots (1)$$

where k is the Thomas angle, μ_o is the angle of flare and B_H, B_V and $\bar{B}$ are the magnetic field intensities for the hill, valley and average fields. In the case of classical cyclotrons, $B_H = B_V$ and $\mu_o = 0$. Since the Thomas angle is given by

$$k = \frac{\pi}{N}\cdot\frac{\bar{B} - B_V}{\bar{B}}\cdot\frac{B_H - \bar{B}}{B_H - B_V} \qquad \ldots\ldots (2)$$

Equation (1) is not applicable to classical cyclotrons as such. In order to avoid this difficulty certain approximations have to be incorporated.

Moreover the Laslett term and the Kerst term have to be calculated and added separately to Equation (1), wherein the spiral angles at the entrance and exit of a hill or a valley have been assumed to be the same.

Jain[2] et al, have derived expressions for ν_z^2 and ν_r^2 for the general case of unequal spiral angles ϵ_1 and ϵ_2. They have assumed the equations

$$\nu_z^2 = -\mu' + F^2 + \sum \frac{a_n'^2 + b_n'^2}{n^2} + \ldots\ldots$$
$$\nu_r^2 = 1 + \mu' + \ldots\ldots \qquad \ldots\ldots (3)$$

derived by Smith and Garren[3] where $\mu' = \frac{r}{\bar{B}}\frac{d\bar{B}}{dr}$ and F^2 is the flutter term.

We have presented a method to compute expressions for ν_z^2 and ν_r^2 from a simplified but generalized approach which was first suggested by Schatz[4] for separated sector cyclotrons.

2. Basic assumptions. - In this approach the magnet of an A.V.F. cyclotron is considered to consist of a series of alternate dipole magnets having field strengths of the hill and valley respectively and the spiral angles at the entrance and exit have been assumed to be unequal both for hills as well for valleys. It is also assumed that for the most generalised case, the hill and valley fields need not be uniform. In other words there is a field index $n = -\frac{r}{B}\frac{dB}{dr}$ both for hills as well as for valleys. The equations of motion can then be set up using the standard transfer matrix for a dipole magnet having a field index and non normal entry and exit angles. The transfer matrix of one period of the magnetic field is then the product of the matrices corresponding to hill sector M_H and valley sector M_V with the usual convention of transfer matrix multiplication for a beam transport system[5].

3. Orbit Equations. - Orbit sections for a hill and valley period have been shown in Figure 1.

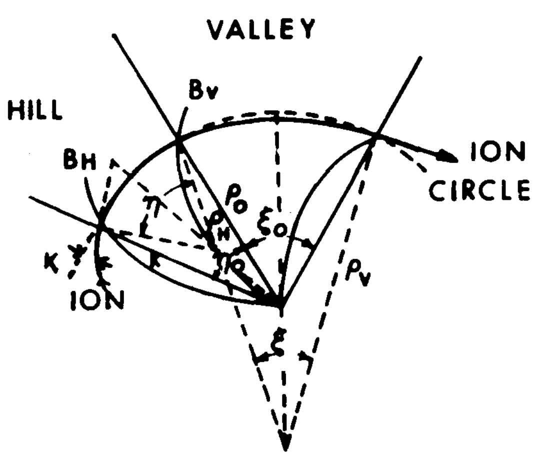

Fig.1 : Section of an equilibrium orbit in the wedge shaped hard edge approximation, consisting of circular arcs of radii of curvature ρ_H and ρ_V corresponding to B_H and B_V respectively.

From simple geometry, it can be shown that the entrance and exit angles to the magnet sectors are given by

$$\beta_1 = \epsilon_1 + \frac{\eta - \eta_0}{2}, \qquad \beta_2 = \epsilon_2 - \frac{\eta - \eta_0}{2} \qquad \ldots (4)$$

From Figure 1,

$$\pi - k - \frac{\eta_0}{2} = \pi - \frac{\eta}{2} \quad \text{or} \quad k = \frac{\eta - \eta_0}{2} \qquad \ldots (5)$$

Hence

$$\beta_1 = \epsilon_1 + k, \quad \beta_2 = \epsilon_2 - k$$

Also one has to keep in mind the sign convention for entry and exit angles, i.e. for a hill β_1 is +ve, β_2 is -ve and for a valley β_2 being the entry angle is +ve and β_1 being the exit angle is -ve.

Following these conventions, the transfer matrix for the hill axial motion $M_{H,Z}$ is a product of three matrices, the non-normal entry dipole, the dipole and the non-normal exit dipole, and is given by

$$M_{H,Z} = \begin{pmatrix} 1 & 0 \\ \frac{\tan\beta_2}{\rho_H} & 1 \end{pmatrix} \begin{pmatrix} \cos(n^{\frac{1}{2}}\phi) & \frac{\rho_0 \sin(n^{\frac{1}{2}}\phi)}{n^{\frac{1}{2}}} \\ -\frac{n^{\frac{1}{2}}\sin(n^{\frac{1}{2}}\phi)}{\rho_0} & \cos(n^{\frac{1}{2}}\phi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan\beta_1}{\rho_H} & 1 \end{pmatrix} \quad \text{where } \phi = \eta \cdot \frac{\rho_H}{\rho_0} \qquad \ldots (6)$$

Similarly the valley axial transfer matrix $M_{V,Z}$ is given by

$$M_{V,Z} = \begin{pmatrix} 1 & 0 \\ \frac{\tan\beta_1}{\rho_V} & 1 \end{pmatrix} \begin{pmatrix} \cos(n^{\frac{1}{2}}\psi) & \frac{\rho_0 \sin(n^{\frac{1}{2}}\psi)}{n^{\frac{1}{2}}} \\ -\frac{n^{\frac{1}{2}}\sin(n^{\frac{1}{2}}\psi)}{\rho_0} & \cos(n^{\frac{1}{2}}\psi) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan\beta_2}{\rho_V} & 1 \end{pmatrix} \quad \text{where } \psi = \xi \cdot \frac{\rho_V}{\rho_0} \qquad \ldots (7)$$

The axial betatron oscillation per turn ν_z is calculated as,

$$\cos\left(\nu_z \cdot \frac{2\pi}{N}\right) = \frac{1}{2} \mathrm{Tr}\left(M_{V,z} \cdot M_{H,z}\right) \qquad \ldots (8)$$

Similarly for the radial betatron oscillations ν_r is given by,

$$\cos\left(\nu_r \cdot \frac{2\pi}{N}\right) = \frac{1}{2} \mathrm{Tr}.\left(M_{V,r} \cdot M_{H,r}\right) \qquad \ldots (9)$$

where,

$$M_{H,r} = \begin{pmatrix} 1 & 0 \\ \frac{\tan\beta_2}{\rho_H} & 1 \end{pmatrix} \begin{pmatrix} \cos[(1-n)^{\frac{1}{2}}\phi] & \frac{\rho_0 \sin[(1-n)^{\frac{1}{2}}\phi]}{(1-n)^{\frac{1}{2}}} \\ -\frac{(1-n)^{\frac{1}{2}}\sin[(1-n)^{\frac{1}{2}}\phi]}{\rho_0} & \cos[(1-n)^{\frac{1}{2}}\phi] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\tan\beta_1}{\rho_H} & 1 \end{pmatrix} \qquad \ldots (10)$$

and

$$M_{V,r} = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\beta_1}{\rho_V} & 1 \end{pmatrix} \begin{pmatrix} \cos[(1-n)^{\frac{1}{2}}\psi] & \frac{\rho_0 \sin[(1-n)^{\frac{1}{2}}\psi]}{(1-n)^{\frac{1}{2}}} \\ -\frac{(1-n)^{\frac{1}{2}}\sin[(1-n)^{\frac{1}{2}}\psi]}{\rho_0} & \cos[(1-n)^{\frac{1}{2}}\psi] \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{\tan\beta_2}{\rho_V} & 1 \end{pmatrix} \qquad \ldots (11)$$

For the purpose of simplicity we have calculated the transfer matrices in the first order. Since these expressions are very involved, it is not convenient to derive any algebraic expression for ν_z and ν_r in the most general case. However in order to show that the expressions reduce to the conventional equations for a radial sector cyclotron we make the substitution $\epsilon_1 = \epsilon_2 = 0$.

Hence

$$\nu_z^2 = n + \frac{N}{\pi} \cdot \tan k \cdot \frac{B_H - B_V}{\bar{B}} - \tan^2 k \frac{(B_H - \bar{B})(\bar{B} - B_V)}{\bar{B}^2} \qquad \ldots(12)$$

$$\nu_r^2 = 1 - n + \tan^2 k \frac{(B_H - \bar{B})(\bar{B} - B_V)}{\bar{B}^2} \qquad \ldots(13)$$

According to Richardson

$$k \sim \frac{\pi}{N} \cdot \frac{\bar{B} - B_V}{\bar{B}} \cdot \frac{B_H - \bar{B}}{B_H - B_V}$$

Hence the second term in Equation (12) reduces to

$$\frac{(B_H - \bar{B}) \cdot (\bar{B} - B_V)}{\bar{B}^2} = F^2 \qquad \ldots(14)$$

and Equations (12) and (13) reduce to

$$\begin{aligned} \nu_z^2 &= n + F^2 - F^2 \tan^2 k \\ \nu_r^2 &= 1 - n + F^2 \tan^2 k \end{aligned} \qquad \ldots(15)$$

These equations can be compared with Equation (3) derived by Smith and Garren and are found to be similar. For $B_H = B_V$, $F^2 = 0$.

Hence Equation (15) reduces to

$$\nu_z^2 = n, \quad \nu_r^2 = (1 - n) \qquad \ldots(16)$$

which is the classical cyclotron equation. In deriving the above equations we have substituted

$$\begin{aligned} \frac{\eta \rho_H + \xi \rho_V}{\rho_0} &= \xi_0 + \eta_0 \\ \xi &= \frac{2\pi}{N} \cdot \frac{B_V}{\bar{B}} \cdot \frac{B_H - \bar{B}}{B_H - B_V} \\ \eta &= \frac{2\pi}{N} \cdot \frac{B_H}{\bar{B}} \cdot \frac{\bar{B} - B_V}{B_H - B_V} \end{aligned} \qquad \ldots(17)$$

For a separated sector case if we substitute $B_V = 0$ and the free space length ℓ is given by

$$\ell = 2 \rho_H \sin\left(\frac{\pi}{N} - \frac{\eta_0}{2}\right) \qquad \ldots(18)$$

where entrance and exit angles are replaced by

$$\beta_1 = \epsilon_1 + \left(\frac{\pi}{N} - \frac{\eta}{2}\right), \quad \beta_2 = \epsilon_2 - \left(\frac{\pi}{N} - \frac{\eta}{2}\right)$$

expressions for ν_z and ν_r become identical with the expressions given by Schatz.

4. Conclusion. - In this simplified approach based on matrix methods, we have been able to derive expressions for ν_z^2 and ν_r^2 which take into account the Thomas focussing, the Laslett contribution as well as the Kerst contribution. It is the most general in the sense that expressions have been derived for unequal spiral angles $\epsilon_1 \neq \epsilon_2$ and $n \neq 0$. Also for the case (1) $\epsilon = 0$ it reduces to the expression for radial sector cyclotron (2) $B_V = 0$ it gives the expression for separated sector cyclotron (3) $B_H = B_V$ one arrives at the classical cyclotron equations.

4. Acknowledgements. - We acknowledge the fruitful discussions we had with S.K. Basu, K. Krishan and A. Bandopadhyay at various stages during our calculations.

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