NEW PROOFS FOR OLD FACTS IN ACCELERATION THEORY

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Abstract.- The linear betatron oscillation is represented by a simple matrix formalism. This method is independent of the orbit geometry. The longitudinal motion connected to the betatron oscillation can easily be treated. It is proved that the change of the orbit due to the acceleration does not affect the revolution time. The general validity of the Hamiltonian describing the phase compression effect is demonstrated.

Introduction

The traditional way to predict the behaviour of beams in cyclic accelerators is to discuss the properties of static equilibrium orbits and to add special corrections for the effects of the acceleration. Recently M. Gordon [1),11)] and W. Schulte [2)] included the acceleration in the basic discussion.

Similar to those just mentioned this work defines an accelerated equilibrium orbit (AEO) in order to approximate the behaviour of a real beam more precisely than a static equilibrium orbit (EO) can do. The essential improvement lies in its ability to describe the longitudinal motion of particles. The results are not new, e.g. the isochronism w.r.t. to tilted acceleration gaps has been known to many people for a long time before Gordon's first official publication [1)]. The method of derivation, however, has definite advantages: it is simple and still very general. It uses a matrix-formalism based on TRANSPORT notation [3)] to describe the oscillations of a particle around its reference orbit. The following assumptions are made:

- the existence of a static equilibrium orbit as a first reference
- the validity of an approximation covering only linear terms
- the symplectic conditions should hold [4),5)]
- the focussing properties should not change substantially between two successive orbits (adiabaticity condition).

In this framework an expression can be derived for the phase-oscillation connected to the betatron-oscillation of a particle. In the definition used here, the accelerated equilibrium orbit has the same revolution time as the static equilibrium orbit. This is proved first for accelerating gaps perpendicular to the orbit and then for inclined gaps.

The phase history of a particle in an accelerator with radially varying energy-gain is described by a Hamiltonian suggested by W.Joho [6)]. A general proof for the validity of this Hamiltonian is given at the end of this paper.

The Definition of the AEO

The first order TRANSPORT-notation [3)] is simplified further by neglecting the motion in x3,x4 (=y,y') and by introducing a short vector- and matrix notation

$$\underline{x}_f = M\,\underline{x}_i + \underline{d}\,\delta = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_i + \begin{pmatrix} R_{16} \\ R_{26} \end{pmatrix} \delta \qquad (1)$$

$$l_f = \underline{a}^T\,\underline{x}_i + l_i = (R_{51}, R_{52}) \begin{pmatrix} x \\ x' \end{pmatrix}_i + l_i \qquad (2)$$

(If the equilibrium orbit of the appropriate energy is taken as a reference one can obtain $\delta=0$). In this formulation the symplectic conditions [4),5)] have the form

$$\underline{d} = \begin{pmatrix} -R_{12} & R_{11} \\ -R_{22} & R_{21} \end{pmatrix} \underline{a} \qquad (3)$$

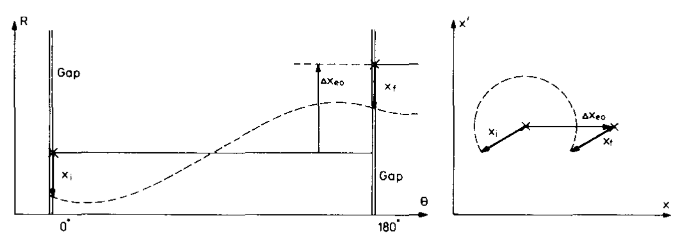

Fig. 1: The definition of the AEO requires the periodicity of the phasespace coordinates w.r.t. to the EO of the appropriate energy. This reference has to be changed after the acceleration.

The definition of the accelerated equilibrium (AEO) orbit is based on a special periodicity condition: after a full cycle of - transformation, acceleration, redefinition of the reference - the phase space coordinates have to be the same (see fig. 1).

$$\underline{x}_i = \underline{x}_f = M\,\underline{x}_i + \Delta\underline{x}_{ac} - \Delta\underline{x}_{eo}$$

The effect of the acceleration on the orbit Δx_{ac} can only contain x'-terms. They are usually small but nevertheless important for the phase history of the particle. The main part, Δx_{eo} is the phase-space difference between two successive static equilibrium orbits separated by the energy gained in the acceleration.

It is now simple to find the starting values in phase space for the accelerated equilibrium orbit: (I = identity matrix)

$$\underline{x}_i = (M - I)^{-1} (\Delta\underline{x}_{eo} - \Delta\underline{x}_{ac}) \tag{4a}$$

Without further derivation the formula for the general case of several gaps is given. (Ni=partial transfermatrix between gaps)

$$\begin{aligned} \underline{x}_i &= (M - I)^{-1}\underline{g}_{eff}\ , \ M = N_n \,..\, N_2\, N_1 \\ \underline{g}_{eff} &= \underline{g}_n + N_n\underline{g}_{n-1} + \,..+ N_n\,..N_2\underline{g}_1 \\ \underline{g}_k &= (\Delta\underline{x}_{eo} - \Delta\underline{x}_{ac})_k \end{aligned} \tag{4b}$$

The condition that (M-I) can be inverted coincides with the need to avoid resonances. (trace(M) $\neq$ 2)

Formula (4) gives a good estimate of where to inject the beam into a ring cyclotron. Only in extreme cases does it need to be improved by terms of higher order.

The Phase-Oscillation

This section is valid for a linear betatron oscillation around any reference orbit. The difference l_n in the longitudinal direction is considered as a function of the initial vector $\underline{x}$ for many crossings of a similar accelerator-section (or repeated crossings of the same interval). From equations (1) and (2) we get

$$\begin{aligned} l_1 &= \underline{a}^T\, \underline{x}_0 + l_0 \\ l_2 &= \underline{a}^T\, \underline{x}_1 + l_1 = \underline{a}^T\, M\, \underline{x}_0 + \underline{a}^T\, \underline{x}_0 + l_0 \\ &\vdots \\ l_n &= \underline{a}^T\, (M^{n-1} + \,..\, + M + I)\, \underline{x}_0 + l_0 \end{aligned}$$

where the long sum of matrices can be transformed with simple algebra to

$$l_n = \underline{a}^T\, (M - I)^{-1}\, (M^n - I)\, \underline{x}_0 + l_0$$

In order to get l_n as a simple function of n, the matrix M is now replaced by its Twiss-representation

$$\begin{aligned} M &= I\cos(\sigma) + J\sin(\sigma),\ 2\cos(\sigma) = \mathrm{trace}(M) \\ M^n &= I\cos(n\sigma) + J\sin(n\sigma),\quad J^2 = -I \end{aligned}$$

This gives the general expression for the longitudinal motion coupled to the betatron oscillation (starting with the vector $\underline{x}_0$):

$$\begin{aligned} l(n) = {} & l_0 - \underline{a}^T\, (M - I)^{-1}\, \underline{x}_0 \\ & + \underline{a}^T (M-I)^{-1} \underline{x}_0 \cos(n\sigma) + \underline{a}^T (M-I)^{-1} J \underline{x}_0 \sin(n\sigma) \end{aligned} \tag{5}$$

The longitudinal motion performs an oscillation centered around $l_0 - \underline{a}^T\, (M - I)^{-1}\, \underline{x}_0$

This center of the oscillation is the relevant phase of the particle in consideration. During the acceleration the effects of positive and negative excursions from this center-value cancel to first order. W. Schulte [10] called the corrected value "center position phase" and A. Chabert [8] and M. Gordon [11] also expressed the need for this correction. Formula (5), which is complete to 1st order is now applied to the smooth approximation model for comparison:

$$M = \begin{pmatrix} \cos(\sigma) & \frac{r}{\nu_r}\sin(\sigma) \\ -\frac{\nu_r}{r}\sin(\sigma) & \cos(\sigma) \end{pmatrix}$$

$$\underline{a}^T = -\left(\frac{1}{\nu_r}\sin(\sigma)\ ,\ \frac{r}{\nu_r^2}\,(1 - \cos(\sigma))\ \right)$$

$$\Delta l = -\frac{r}{\nu_r^2}\, x_0'\quad ,\ \Delta\phi = \frac{2\pi}{2\pi}\frac{h}{r}\,\Delta l = -\frac{h}{\nu_r^2}\, x_0'$$

This expression agrees with the results in the papers quoted above, assuming ν_r = 1 for [8] and [10]

The Isochronism of the AEO

The AEO has the same revolution-time as the static equilibrium orbit. This will be proved in two parts.

For the first part, Δx_{ac} is assumed to be zero, i.e. perpendicular crossing of the accelerating gaps is requested and no radial gradient of the energy gain is allowed.

In order to be able to calculate Δx_{eo}, the adiabaticity condition has to be introduced: it requires that the subsequent equilibrium orbit can be approximated as a linear motion relative to the original EO (1). This relative motion must of course be periodic:

$$\begin{aligned} \Delta\underline{x}_{eo\ i} = \Delta\underline{x}_{eo\ f} &= M\ \underline{x}_{eo\ i} + \underline{d}\ \delta \\ \Delta\underline{x}_{eo} &= -(M - I)^{-1}\ \underline{d}\ \delta \end{aligned} \tag{6}$$

Using equation (4a) the accelerated equilibrium orbit has the starting value.

$$\underline{x}_i = -(M - I)^{-2}\ \underline{d}\ \delta$$

The difference in length between the two orbits is given by formula (2) and the symplectic condition is used to express $\underline{d}$ in terms of $\underline{a}$

$$\begin{aligned} \Delta l &= -\underline{a}^T\ (M - I)^{-2}\ \underline{d}\ \delta \\ &= -\underline{a}^T\ (M - I)^{-2} \begin{pmatrix} -R_{12} & R_{11} \\ -R_{22} & R_{21} \end{pmatrix} \underline{a}\ \delta \\ \Delta l &= -\underline{a}^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{a}\ \frac{\delta}{\det(M-I)} = 0 \qquad \text{qed} \end{aligned}$$

This result is general for all gaps in the case of several gaps. The difference in length between each partial oscillation and the EO is zero.

The second part of the proof considers only Δx_{ac} from oblique crossings of the gap and again neglects radial voltage gradients. The additional oscillation on top of the one treated in the first part is considered using eqn. (4). The difference in length (eqn. 2) for such an oscillation around the static reference becomes

$$l_1 - l_0 = \underline{a}^T (M - I)^{-1} \Delta \underline{x}_{ac}$$

The usage of the symplectic conditions and of the fact that Δx_{ac} has only x'-terms yields:

$$l_1 - l_0 = ((1 - R_{22}),\ R_{12}),\ \underline{d}\ \frac{\Delta x'_{ac}}{\det(M-I)}$$

The acceleration acts only perpendicular to the gap. The parallel part of the momentum p_{II} remains constant.
The derivative of $\sin(w) = p_{II}/p$ gives:

$$\cos(w)\ \Delta w = -\frac{p_{II}}{p^2}\ \Delta p = -\sin(w)\ \frac{\Delta p}{p}$$

$$\Delta x'_{ac} = \Delta w = -\tan(w)\ \delta$$

$$l_1 - l_0 = -((1-R_{22}),\ R_{12})\ \underline{d}\ \frac{\tan(w)\ \delta}{\det(M-I)}$$

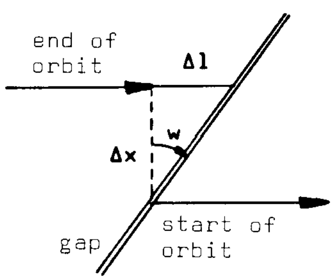

Fig. 2: The additional length Δl which is needed to reach the inclined gap after a radial gain of Δx is simply

$$\Delta l = \Delta x\ \tan(w)$$

This Δx is the first component of $\Delta \underline{x}_{eo}$ from (6)

$$\Delta x = (\Delta \underline{x}_{eo})_x = (-(M - I)^{-1}\ \underline{d}\ \ \delta)_x$$

$$\Delta l = -((R_{22}-1),\ -R_{12})\ \underline{d}\ \frac{\delta\ \tan(w)}{\det(M-I)}$$

$$\Delta l = l_1 - l_0 \qquad \text{qed} \qquad (7)$$

The revolution time of the accelerated orbit is shifted just the right amount to reach the tilted gap again in the same time as the static orbit needs for the period between fixed azimuths.

The extension to several gaps is not trivial. If the gaps have different inclinations, the phase history is a weighted mixture of the different contributions. It might not fit anymore to any single gap.

The Hamiltonian of the Phase-Compression

In this section one inclined accelerating gap (angle=w) per period is considered, having a finite thickness described by the transit angle α and a radial gradient of the voltage Vpk. Uniform voltage distribution is assumed across the gap. In such a system the energy-gain per period and the Hamiltonian describing the phase compression effect [6] are:

$$\frac{dE}{dn} = q\ Vpk\ \frac{\sin(\alpha/2)}{(\alpha/2)}\ \cos(\phi) = \frac{\partial H}{\partial \phi}$$

$$H = q\ Vpk\ \frac{\sin(\alpha/2)}{(\alpha/2)}\ \sin(\phi)$$

Now $d\phi/dn$ is calculated using the formulas from the above sections. The first part is due to the RF-magnetic field

$$\frac{d\phi_1}{dn} = -\frac{\omega}{v}\ \underline{a}^T\ (M - I)^{-1} \binom{0}{\Delta x'}$$

$$\underline{a}^T\ (M-I)^{-1}\binom{0}{1} = \frac{-1}{\det(M-I)}((1-R_{22}),R_{12})\ \underline{d}$$

$$\frac{db}{dt} = \mathrm{rot}(\varepsilon) = -\frac{\partial Vpk}{\partial x}\ \frac{\cos(w)}{g}\ \cos(\phi)$$

$$\Delta x' = \int b\ ds\ \ /\ p$$

$$\Delta x' = -\frac{\partial Vpk}{\partial x}\ \frac{\cos(w)}{g\ \omega\ p/q}\ \int_{gap} \sin(\phi^*)\ ds$$

$$= \frac{-\partial Vpk/\partial E}{dx/dp\ dp/dE}\ \frac{q}{p\ \omega}\ \frac{\sin(\alpha/2)}{(\alpha/2)}\ \sin(\phi)$$

$$\frac{dx}{dp}\ \frac{dp}{dE} = \frac{1}{\det(M-I)}((R_{22}-1),-R_{12})\ \underline{d}\ \frac{1}{p}\ \frac{1}{v}$$

$$\frac{d\phi_1}{dn} = -\frac{\omega}{v}\ \frac{\partial Vpk/\partial E}{1/p\ \ 1/v}\ \frac{q}{p\ \omega}\ \frac{\sin(\alpha/2)}{(\alpha/2)}\ \sin(\phi)$$

$$\frac{d\phi_1}{dn} = -q\ \frac{\partial Vpk}{\partial E}\ \frac{\sin(\alpha/2)}{(\alpha/2)}\ \sin(\phi)$$

For the second part, the inclination of the gap center line can be neglected according to (7), but the relative tilt of a line at phase ϕ^* within the gap would ask for a contribution to $d\phi/dn$.

$$\left.\frac{d\phi}{dn}\right|_{\phi^*} = \frac{\phi - \phi^*}{\alpha}\ \frac{d\alpha}{dE}\ \frac{dE}{dn}$$

The total $d\phi/dn$ from the transittime effect is a weighted average over the range $\phi \pm \alpha/2$

$$\frac{d\phi_2}{dn} = \frac{dE}{dn}\ \frac{\int (\phi-\phi^*)\ \cos(\phi^*)\ d\phi^*}{\int \cos(\phi^*)\ d\phi^*}\ \frac{1}{\alpha}\ \frac{d\alpha}{dE}$$

$$\frac{d\phi_2}{dn} = q\ Vpk\ \sin(\phi)\frac{\sin(\alpha/2)-(\alpha/2)\cos(\alpha/2)}{(\alpha/2)^2}\ \frac{d(\alpha/2)}{dE}$$

As it is required from theory the two parts together agree with the partial derivative of the Hamiltonian:

$$\frac{d\phi}{dn} = \frac{d\phi_1}{dn} + \frac{d\phi_2}{dn} = -\frac{\partial H}{\partial E}$$

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