TRANSFER MATRIX TECHNIQUES FOR RING CYCLOTRON SECTOR SHAPES

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Summary

A transfer matrix technique has been developed for obtaining the sector shape of high energy cyclotron magnets and has been employed in the zero-order design of the 3 GeV and 15 GeV superconducting ring cyclotrons for the proposed CANUCK kaon factory. Effects taken into account include return flux gullies across the orbit region and also soft magnet edges. The valley field may be non-zero with polarity equal or opposite to that of the hill. This technique has been checked by running a synchrotron lattice code (DIMAT) on the predicted shape at selected energies. The method is applicable both to an isochronous cyclotron and to an FFAG accelerator in which the isochronous condition is abandoned. The code (RING) which performs the functions mentioned supplies Fourier coefficients of the magnetic field to allow for comparisons of orbit dynamic quantities with those of general orbit theories.

Introduction

For the proposed TRIUMF kaon factory[1] one accelerator option (CANUCK - Canadian University Cyclotrons for Kaons) consists of two high energy superconducting isochronous ring cyclotrons, capable of accelerating 100 μA of protons to 15 GeV. In order to obtain a first sector shape, a matrix method has been applied. This shape was then used in a field calculating code for orbit tracking and further sector shimming.[2,3] In this paper we discuss the applied matrix method, and extra features introduced in it for superconducting ring cyclotrons.

Hard edge matrix expressions for ring cyclotrons with field free valleys have been given by Schatz[4] and Gordon.[5] In superconducting ring cyclotrons the return flux in the valleys may amount to fields of -1T.[2,6] For this type of cyclotron this paper describes the method used to obtain the spiral angle necessary for a specified value of ν_z, and consequently to obtain the entire sector shape. Although we outline it for constant fields, a local field index may be introduced. Soft edges between hill and valley are taken into account via the K_1 and K_2 constants of TRANSPORT[7] and affect only the vertical motion, tending to reduce ν_z. Richardson[8] has proposed the use of return flux gullies alongside the hills, realized by having part of the return-yoke across the orbit region with a gap in the median plane for passage of the ions. This would increase the flutter, thus reducing the maximum spiral angle, and provide field free regions between the sectors useful for superconducting cavities. For this type of machine the matrix method has been extended, leading to expressions for ν_r and ν_z dependent on the 4th power of the spiral angle.

Constant Field Valleys

We first consider the simplified case of constant positive field hills and constant negative field valleys. Figure 1 gives the geometry and the definition of some geometrical quantities. The bending angle in hill and valley are θ_H and θ_V, path lengths ℓ_H and ℓ_V, radii ρ_H and ρ_V. Closure of orbits imposes two conditions on the geometry of the cyclotron:

$$\ell_H + \ell_V = 2\pi\bar{R}/N$$
$$\ell_H/\rho_H - \ell_V/\rho_V = 2\pi/N \qquad (1)$$

where N is the number of sectors, $\bar{R} = \beta R_c$ is the average radius. For positive field valleys the sign in the second of eqs. (1) is positive. For an isochronous machine $R_c = c/\omega$ is constant, where ω is the particle revolution frequency. Hence the average field is $\bar{B} = \gamma B_o$ with $B_o = m_o\omega/q$ constant. For an FFAG accelerator with varying RF ("ring-synchrocyclotron") the quantities R_c and B_o depend on the prescribed frequency curve $\omega=\omega(t)$.

Defining $\alpha = B_V/B_H$, so that $\rho_V^{-1} = |\alpha|\rho_H^{-1}$, gives:

$$\ell_H = (2\pi/N)(\rho_H-\alpha\bar{R})/(1-\alpha) \ . \qquad (2)$$

With this all lengths and angles from Fig. 1 are determined, in particular:

$$\begin{aligned} a &= \rho_V \sin \theta_V/2 \\ b &= \rho_H \cos \theta_V/2 + (a\pm\rho_H\sin\theta_V/2)\cot \pi/N \\ \theta &= \tan^{-1}(a/b) \\ \theta' &= \pi/N - \theta \\ \kappa &= \theta \pm \theta_V/2 \\ r_H &= (a^2+b^2)^{1/2} \end{aligned} \qquad (3)$$

(+ in the case of negative, - for positive field valleys, e.g. for TRIUMF, compact AVF cyclotron, etc.; for α=0, $a=\ell_V$).

Thus for each selected energy E, or $\bar{R}$, or r_H the geometry is fixed. For an isochronous machine, as well as in general for any FFAG, there will be flaring of the hill given by:

$$\tan \mu = r_H \ \partial\theta'/\partial r_H \ . \qquad (4)$$

Furthermore in order to maintain axial focusing in general the hill will be spiralled, the spiral angle θ_{sp} being given by:

$$\tan \varepsilon = r_H \ \partial\theta_{sp}/\partial r_H \ . \qquad (5)$$

The angles μ and ε are related to the angles γ_f and γ_d at the hill boundary[9]:

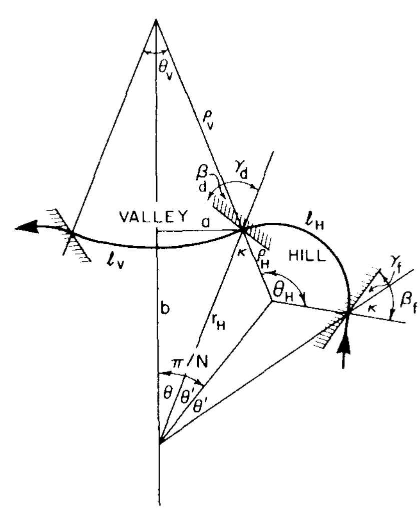

Fig. 1. Ring cyclotron with constant negative field valleys.

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$$\tan\gamma_f = \tan\varepsilon - \tan\mu$$
$$\tan\gamma_d = \tan\varepsilon + \tan\mu \,. \qquad (6)$$

As will be seen, the angle ε is fixed by choosing the value of ν_z. This then fixes the geometry of the entire cyclotron sector for all energies.

The vertical tune ν_z is evaluated from the transfer matrix M_z:

$$M_z = \begin{pmatrix}1 & \ell_V\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -F_d & 1\end{pmatrix}\begin{pmatrix}1 & \ell_H\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -F_f & 1\end{pmatrix} \qquad (7)$$

where: $F_f = \rho_{HV}^{-1}\tan\beta_f$

$$F_d = -\rho_{HV}^{-1}\tan\beta_d$$
$$\rho_{HV} = \rho_H(1-\alpha)$$
$$\beta_f = \gamma_f + \kappa$$
$$\beta_d = \gamma_d - \kappa \qquad (8)$$

Hence one obtains:

$$2\cos(2\pi\nu_z/N) = 2 - (F_f+F_d)(\ell_H+\ell_V) + F_fF_d\ell_H\ell_V$$
$$= 2 - 2(\ell_H+\ell_V)\rho_{HV}^{-1}\,\frac{t_nt_d+\tan^2\varepsilon\,\tan\kappa}{t_d^2-\tan^2\varepsilon\,\tan^2\kappa}$$
$$+\ \ell_H\ell_V\rho_{HV}^{-2}\,\frac{\tan^2\varepsilon - t_n^2}{t_d^2-\tan^2\varepsilon\,\tan^2\kappa}\,, \qquad (9)$$

with:
$$t_n = \tan\kappa - \tan\mu$$
$$t_d = 1 + \tan\kappa\,\tan\mu \,. \qquad (10)$$

In this expression the spiral angle appears only as $\tan^2\varepsilon$. Hence for a predescribed value of ν_z eq. (9) can be solved for $\tan^2\varepsilon$. By integration of eq. (5) the entire sector shape is now fixed. For each energy the edge focusing angles β_f and β_d are known as well as the magnet lengths so that, in turn, a direct matrix multiplication may be carried out yielding ν_r and ν_z. Moreover the Twiss parameters[10] are known, in particular the envelope functions β_r, β_z and the dispersion function η through one period.

A 3.5 GeV Superconducting Cyclotron

Table I gives some parameters of the first and second stage cyclotrons of the CANUCK kaon factory.[1] The sector shape obtained was used to define the coil shape around the iron pole pieces, and the magnetic field was calculated using Biot and Savart's law.[2,3])

Table I
CANUCK I & II RING Specifications

Energy range	430 MeV - 3.5 GeV	3.5 GeV-1.5 GeV
# sectors	15	42
Hill field	4.5 T	6T
Valley field	-0.5 T	-1T
Radial range	7.5 m - 10.1 m	40.5m-41.3m
RF frequency	46 MHz	115 MHz
Harmonic	10	100
Tune ν_z	3.3	3.3
Tune ν_r	1.5 - 5.2	4.2-18.3
Max. spiral angle	70°	81°

Table II gives some data for the 3.5 GeV cyclotron, obtained with our code RING which does the calculations outlined in the previous section. Figure 2 shows the sector shape obtained. We also ran a second order synchrotron lattice code (DIMAT[11]) for selected energies on the cyclotron. As an example Fig. 3 gives the lattice functions for one period (one sector) for E=2 GeV. The radial beam width is given by $(\beta_rE_r)^{1/2}$ with E_r the radial emittance in π mm mrad; similarly for the vertical beam width. The tunes correspond to the RING values. The transition energy is equal to the particle energy, demonstrating isochronism. The dispersion function η corresponds to the distance of the periodic orbit of off-momentum particles with respect to the 2 GeV orbit given in Table II. It changes sign in the valley because of the reversed field there. The chromaticity $\xi_r = \partial\nu_r/\partial(\Delta p/p) = 0.178$, $\xi_z = 0$ (RING values), for which DIMAT requires a curvature of the left and right hill edges with radius -1.572 m and +1.549 m respectively. This corresponds to the RING-calculated curvature (Fig. 2).

Table II
Characteristics of the 3.5 GeV cyclotron

E [GeV]	$\bar{R}$ [m]	B [T]	ℓ_H [m]	ℓ_V [m]	β_f [deg]	β_d [deg]	ν_r	ν_z
0.5	7.84	0.46	0.70	2.58	17.98	+17.98	1.62	4.17
1	9.06	0.63	0.95	2.85	32.91	- 9.73	2.15	3.30
1.5	9.55	0.79	1.14	2.86	44.71	-31.24	2.70	3.30
2	9.81	0.95	1.32	2.79	51.82	-45.47	3.27	3.30
2.5	9.96	1.11	1.49	2.68	56.95	-55.48	3.86	3.30
3	10.05	1.27	1.66	2.56	60.89	-62.62	4.47	3.30
3.5	10.12	1.43	1.82	2.42	64.04	-67.68	5.11	3.30

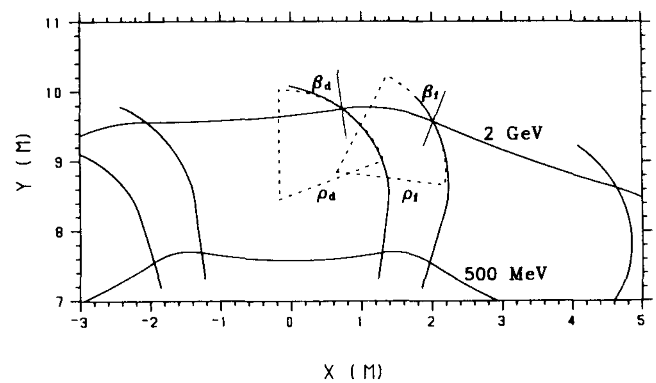

Fig. 2. Sector shape for the 3.5 GeV cyclotron with the 500 MeV and 2 GeV orbits. Curvature radii ρ_f = 1.549 m and ρ_d = 1.572 m, computed by DIMAT, have been indicated for 2 GeV. Table II gives some data for this cyclotron.

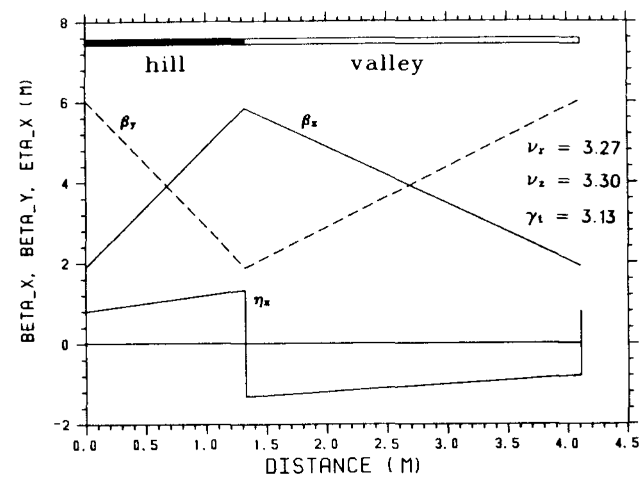

Fig. 3. Envelope and dispersion function for the 2 GeV orbit calculated with DIMAT.

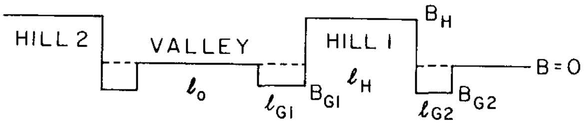

Fig. 4. Field profile for a cyclotron with return flux gullies.

Cyclotrons with Return Flux Gullies

We now consider the case where localized return flux gullies[8] have been provided alongside the hill to enhance the flutter, and reduce the spiral angle requirement. Figure 4 shows a field profile. The gullies are parametrized by:

$$B_{Gi} = -\alpha_i B_H$$
$$\ell_{Gi} = \eta_i \ \ell_H \qquad (11)$$

for i = 1,2. The flutter is given by:

$$F^2 = (\bar{R}/\rho_H)(1+\alpha_1^2\eta_1+\alpha_2^2\eta_2)/(1-\alpha_1\eta_1-\alpha_2\eta_2)-1 \ . \qquad (12)$$

Figure 5 shows the geometry for a machine with gullies. Index 1 refers always to the defocusing side of the hill (for axial motion), 2 to the focusing side. As in eq. (1) one now obtains the path lengths ℓ_H, ℓ_{Gi}, $\ell_o = \ell_1 + \ell_2$ in terms of the parameters α_i and η_i. This fixes the entire geometry for each energy. In order to evaluate the angles ϕ_i and θ_i, the lengths ℓ_1 and ℓ_2 and b need to be known. This is achieved by the observation that $\phi_1 + \phi_2 = 2\pi/N$ and $r_1 = r_2$. The lengths $\ell'_o = \ell'_1 + \ell'_2$ and $\Delta b = b_1 - b_2$ can immediately be expressed in terms of the parameters α_i and η_i; for ℓ'_1 and b_1 one then finds:

$$2\ell'_1 = \ell'_o - \Delta b \sin(2\pi/N)/(1-\cos 2\pi/N)$$
$$b_1 = (\ell'_2+\ell'_1 \cos 2\pi/N)/\sin 2\pi/N \ . \qquad (13)$$

With eq. (13) all lengths and angles are known, in particular the radii r_{Hi}, r_{Gi} and the angles θ_i, θ'_i, ϕ_i and ϕ'_i.

The flare of the sector is now given by:

$$\tan \mu_i = r_{Hi} \ \partial\theta_i/\partial r_{Hi}$$
$$\tan \mu_{Gi} = r_{Gi} \ \partial\phi'_i/\partial r_{Gi} \qquad (14)$$

The spiral angle θ_{sp} of the sector is a function of the energy. We define $\tan \varepsilon \equiv \bar{R} \ \partial\theta_{sp}/\partial\bar{R}$. The effect of the spiralling at other radii (R_{Hi}, R_{Gi}) belonging to the same energy, is different. Hence the angles γ at the hill and gully boundary are given by:

$$\tan \gamma_f = \alpha_{H2} \tan \varepsilon - \tan \mu_2$$
$$\tan \gamma_d = \alpha_{H1} \tan \varepsilon + \tan \mu_1$$
$$\tan \gamma_1 = \alpha_{G1} \tan \varepsilon + \tan \mu_{G1}$$
$$\tan \gamma_2 = \alpha_{G2} \tan \varepsilon - \tan \mu_{G2}$$

where:

$$\alpha_{Hi} = (r_{Hi}/\bar{R})(\partial\bar{R}/\partial r_{Hi}) \ , \text{ etc.} \qquad (15)$$

The edge focusing angles are given by:

$$\beta_f = \gamma_f + \kappa_2$$
$$\beta_d = \gamma_d - \kappa_1$$
$$\beta_1 = \gamma_1 - \phi_1$$
$$\beta_2 = \gamma_2 + \phi_2 \qquad (16)$$

For soft edge calculations of axial motion these are reduced by effective angles ψ, expressed in terms of the K_1, K_2 constants of K. Brown[7] or the I-parameters of Enge.[12]

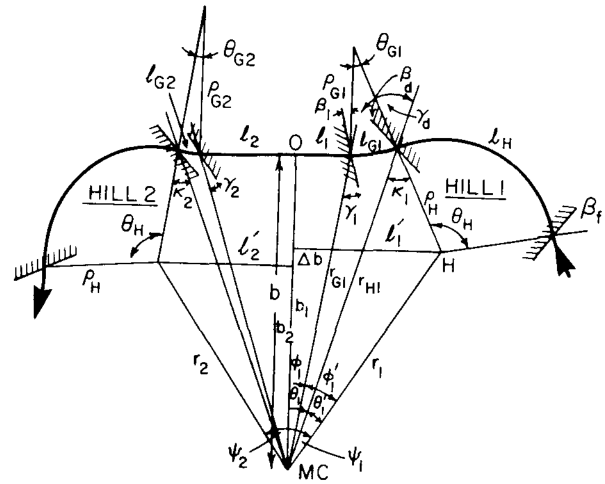

Fig. 5. Geometry for a cyclotron with return flux gullies.

To obtain an expression for ν_z in terms of $\tan \varepsilon$ we proceed as in section 2. The vertical transfer matrix for one period is given by:

$$M_z = \begin{pmatrix} 1 & 0 \\ -F_f & 1 \end{pmatrix} \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F_d & 1 \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_3 & h_1 \end{pmatrix} \qquad (17)$$

where the "h" matrix represents the matrix for the hill (with field gradients if present) and where the "v" matrix is given by:

$$\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} = \begin{pmatrix} c_2 & a_2 \\ b_2 & c_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \ell_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -f_1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & a_1 \\ b_1 & c_1 \end{pmatrix} \qquad (18)$$

where the first and last matrices in the product represent the gullies. The focal strengths F and f contain the edge focusing angles β, e.g.:

$$F_f = (\rho_H^{-1} + \rho_{G2}^{-1}) \tan(\beta_f - \psi_f) \ . \qquad (19)$$

The expansion of $Tr(M_z)$, similar to that in eq. (9), in the powers of $\tan \varepsilon$ is given in ref. 13. One arrives at the expression:

$$\cos 2\pi\nu_z/N = \sum_{k=0}^{4} \lambda_k \tan^k\varepsilon \Big/ \sum_{k=0}^{4} \pi_k \tan^k\varepsilon \ , \qquad (20)$$

or, with prescribed value of ν_z,

$$\sum_{k=0}^{4} \gamma_k \tan^k\varepsilon = 0 \ . \qquad (21)$$

RING calculates the coefficients γ_k and finds $\tan \varepsilon$ as the smallest positive root. The spiral function θ_{sp} is again found by integration of $\tan \varepsilon$.

For equal parameters $\alpha_1 = \alpha_2$ and $\eta_1 = \eta_2$ (symmetric gullies) only the k = 0,2,4 terms appear in eq. (21), demonstrating that sector spiralling may be directed clockwise or anticlockwise. For asymmetric gullies the k=1 and 3 terms are, in general, also present. We note that this is not in contradiction with time reversal invariance.

A Superferric 0.2 to 1.2 GeV FFAG

As a design example a superferric FFAG in the 1 GeV range with return flux gullies will be discussed. The field in the hill is taken as 3 T and in the return yoke across the orbit area as -2 T. The gully path lengths are 0.25 (left) and 0.35 (right) of that of the hill. The vertical tune is ν_z = 3.7. The RF frequency swing is chosen to give an almost constant ν_r (scaling). Table III gives some beam dynamics quantities computed by RING. The maximum spiral angle is 57.2°.

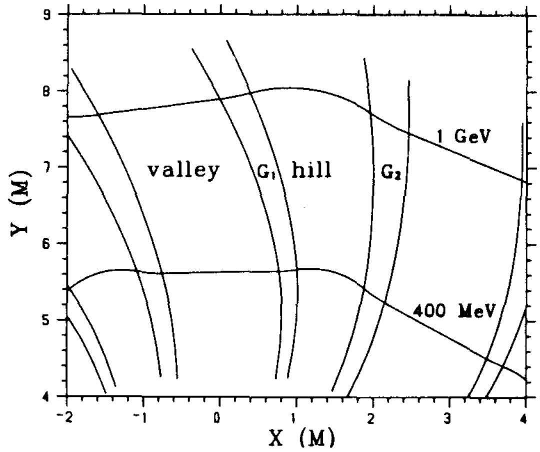

Fig. 6. Sector shape for the superferric FFAG. Data are given in Table III.

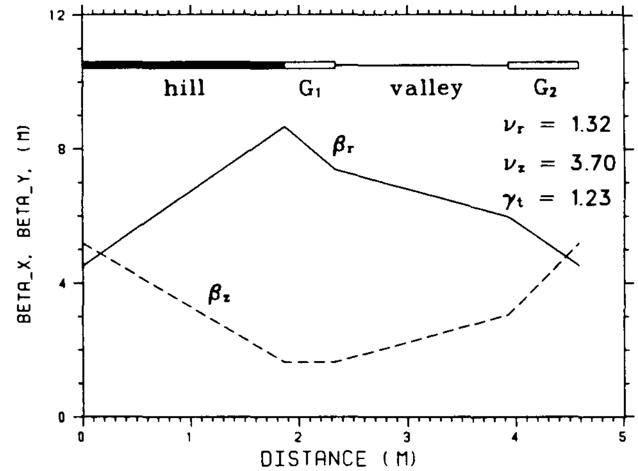

Fig. 7. Envelope functions for the 1.2 GeV orbit in the FFAG.

This is 17° less than in the case without gullies. Figure 6 shows the sector shape. Figure 7 gives the lattice functions for E = 1.2 GeV, showing thatthe gullies essentially provide doublet focusing, giving small beam envelopes in the valleys.

General Remarks and Conclusions

- RING gives the Fourier coefficients of the field for the predicted sector shape, allowing comparisons to be made with orbit dynamics quantities predicted by general orbit theories.[14]
- TRANSPORT has been applied at selected energies for obtaining sector shapes of ring cyclotrons with positive or negative valley fields though not with gullies.[15]

A transfer matrix method has been described for cyclotrons with various degrees of complexity. It provides the sector shape for prescribed values of ν_z. RING calculates the quantities mentioned, and is self-consistent, since it does a direct matrix calculation on the predicted shape, yielding ν_r and the prescribed value of ν_z. In this respect it is equivalent to a synchrotron lattice code, also supplying envelope functions, dispersion, and other quantities.

Table III
Data for a superferric FFAG

E [GeV]	$\overline{R}$ [m]	$\overline{B}$ [T]	Freq [MHz]	ν_r	ν_z	tan ε
0.200	4.357	0.493	6.2000	1.211	3.829	0.000
0.400	5.728	0.556	5.9400	1.320	3.700	0.255
0.600	6.657	0.611	5.6800	1.379	3.700	0.483
0.800	7.411	0.659	5.4200	1.390	3.700	0.750
1.000	8.091	0.699	5.1600	1.364	3.700	1.069
1.200	8.750	0.732	4.9000	1.325	3.700	1.552

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