A CYCLOTRON ORBIT CODE USING SECOND ORDER TRANSFER MATRICES[*]

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Abstract

Programs using second order transfer matrices have been in use for many years in both beamline and synchrotron calculations, but the writers of cyclotron oriented codes have tended to find first order equations sufficient[1,2]. The small magnet gap and tight spiral of the MSU superconducting cyclotrons make the second order effects significantly larger than in conventional machines, so it was deemed that second order matrices would be necessary to obtain reasonable accuracy. The program SOMA, (Second Order Matrix Approximation), has been written to check the validity of this assumption. The agreement between SOMA and orbit integration routines is discussed in the context of a realistic magnetic field for the K500 cyclotron. Also presented is a technique for treating accelerating gaps which have a generalized spiral shape.

Introduction

For many years computer programs for the design of charged particle transport systems have made use of a matrix algebra formalism. The procedure is based on the fact that to first order the final conditions may be expressed as simple integrals of a few particular first order trajectories (matrix elements) characterizing a system. In these codes; beamline elements are represented by idealized components for which the trajectories were derived analytically. The programs then compute a transfer matrix for the whole system by multiplying together the transfer matrices for each of the elements in the system. In a procedure described by K. Brown[3], this technique was generalized to include second order effects in the very successful program "TRANSPORT". The extension to cyclotrons is complicated by the fact that the beam path does not consist of a set of discrete single function elements, but rather a single, very complex magnetic field, which varies as a function of radius and therefore as a function of energy. The well known solution to this was to compute a set of trajectories around a closed (equilibrium) orbit for a set of energies spanning the range of the cyclotron. Then the initial conditions are multiplied by the appropriate matrix to determine the orbit parameters at the first accelerating gap. At the gap a delta function model is used to evaluate the energy gain of the particle. The conditions at this gap can then be multiplied by the appropriate matrix to determine the conditions at the next gap, and so on. The actual matrix coefficients used are determined by interpolating between the values that were stored for the discrete set of energies.

Theory

Let the displacement of a particle from its equilibrium orbit at a given angle, θ, be;

$$x = r - r_{eo} \ , \quad p_x = p_r - p_{reo} \ , \quad \chi = \tau - \tau_{eo} \ , \tag{1}$$

where r and p_r are the radial coordinates of the particle and τ is the time. SOMA assumes that the particle motion can be expressed as a second order Taylor expansion of the initial conditions[3]. Thus the values of the displacements at an azimuth θ, are;

$$\begin{pmatrix} x \\ p_x \end{pmatrix}_\theta = X(\theta,\theta_i) \begin{pmatrix} x \\ p_x \end{pmatrix}_{\theta_i} + A(\theta,\theta_i)\, V_x(\theta_i),$$

$$\begin{pmatrix} z \\ p_z \end{pmatrix}_\theta = Z(\theta,\theta_i) \begin{pmatrix} z \\ p_z \end{pmatrix}_{\theta_i} + D(\theta,\theta_i)\, V_z(\theta_i), \tag{2}$$

$$\chi(\theta) = \chi_1 x(\theta_i) + \chi_2\, p_x(\theta_i) + E(\theta,\theta_i)\, V_x(\theta_i),$$

where the linear transfer matrices are X and Z , and the second order transfer matrices are A, D and E. The vectors $V_x(\theta)$ and $V_z(\theta)$ are,

$$V_x(\theta) = \begin{bmatrix} x^2 \\ x\,p_x \\ p_x^2 \\ z^2 \\ z\,p_z \\ p_z^2 \end{bmatrix}_\theta \qquad V_z(\theta) = \begin{bmatrix} x\,z \\ x\,p_z \\ p_x\,z \\ p_x\,p_z \end{bmatrix}_\theta .$$

The first order transfer matrices are obtained by integrating a set of linearized equations along the EO for two sets of initial conditions. Using the x motion as an example,

$$X(\theta,\theta_i) = \begin{pmatrix} x_1 & x_2 \\ p_{x1} & p_{x2} \end{pmatrix}_{\theta_f} \ , \quad \text{when} \ ,$$

$$x_1(\theta_i)=1, \ p_{x1}(\theta_i)=0, \ x_2(\theta_i)=0, \ \text{and} \ p_x(\theta_i)=1.$$

After the differential equations of motions are expanded to second order in the displacements from the EO, they can be written in the following form;

$$\frac{d}{d\theta}\begin{pmatrix} x \\ p_x \end{pmatrix}_\theta = K(\theta)\begin{pmatrix} x \\ p_x \end{pmatrix}_\theta + \alpha(\theta)\, V_x(\theta),$$

$$\frac{d}{d\theta}\begin{pmatrix} z \\ p_z \end{pmatrix}_\theta = L(\theta)\begin{pmatrix} z \\ p_z \end{pmatrix}_\theta + \beta(\theta)\, V_z(\theta), \tag{3}$$

$$\frac{d\chi(\theta)}{d\theta} = m_1 x(\theta) + m_2 p_x(\theta) + \gamma(\theta)\, V_x(\theta),$$

where the matrices α, β and γ can be obtained from an expansion of Hamilton's equations. The expansions for the displacements found in equations (2) are substituted into the differential equations (3), and terms up to second order in the displacements are retained. Collecting the coefficients of the initial values of the displacements leads to a differential equation for each expansion coefficient. These equations show a very systematic pattern;

$$\frac{d}{d\theta} X = K(\theta)\, X \qquad \frac{d}{d\theta} Z = L(\theta)\, Z \tag{4}$$

$$\frac{d}{d\theta} A_n = K(\theta) \, A_n + F_n(\theta)$$

$$\frac{d}{d\theta} D_n = L(\theta) \, D_n + G_n(\theta) \qquad (5)$$

$$F_n(\theta) = \begin{pmatrix} f_{1n}(\theta) \\ f_{2n}(\theta) \end{pmatrix} \qquad G_n(\theta) = \begin{pmatrix} g_{1n}(\theta) \\ g_{2n}(\theta) \end{pmatrix}$$

$$\frac{d}{d\theta} \gamma_n = m_1 \, a_{1n} + m_2 \, a_{2n} + h_n \qquad , \qquad (6)$$

where the f's are functions of the first order coefficients (the elements of X) and the elements of α. The same is true for the g's and the h's. Inspection of equation (5) shows that it is very similar in form to equation (4) except for the presence of the driving term (F_n or G_n). Since the driving terms are only functions of the first order expansion coefficients, the solutions to equations (5) can be found using a Green's function. For example;

$$A_n(\theta) = \int_o^\theta F_n(\theta') \, \Lambda(\theta,\theta') \, d\theta' \quad ,$$

where the Green's function Λ is [4],

$$\Lambda(\theta,\theta') = X(\theta) \, X^{-1}(\theta') \quad .$$

Crossing the Acceleration Gaps

In the superconducting cyclotrons at MSU the accelerating gaps are of a spiral shape and so the gap position is a function of radius. When the equations of the matrix elements are integrated along the equilibrium orbit they start and stop on radial lines that pass through the point where the EO crosses the gap. The result of this is that when the displaced rays are transferred up to the gap, the values for the displacements (x, p_x etc.) are the values along a radial line. In order to compute the effects of the acceleration correctly, the values of r, p_r and τ are needed at the point the displaced orbit crosses the gap. Since the values of the orbit at this point are not known, they must be estimated. The angle of the gap $\theta_g(r)$ is input as a table of θ values and the program uses a double three point Lagrangian interpolation to find the first and second order derivatives;

$$\frac{d}{dr} \theta_g = \alpha_1 \quad , \text{ and } \quad \frac{d^2}{dr^2} \theta_g = \alpha_2 \quad .$$

The difficulty arises because as the angle of the gap location is corrected for the displacement from the EO, the value of the displacement changes. It can be shown[5] that the correction, $\delta\theta$, to the gap position and δr to the radius are;

$$\delta r \approx x + (r_0' + x')\delta\theta_1 + r_0' \, \delta\theta_2 + \tfrac{1}{2} r_0'' \, \delta\theta_1^2 \quad ,$$

$$\delta\theta = \delta\theta_1 + \delta\theta_2$$

$$\delta\theta_1 = \frac{x \, \alpha_1}{1 - r_0' \, \alpha}$$

$$\delta\theta_2 = \frac{(x' \, \delta\theta_1 + \tfrac{1}{2} r_0'' \, \delta\theta_1^2) \, \alpha_1 + \tfrac{1}{2} \delta r_1^2 \, \alpha_2}{1 - r_0' \, \alpha_1} \quad ,$$

where the prime means differentiation with respect to θ, the zero subscript indicates the EO value, and the subscript '1' refers to the first order component. The procedure for calculating the changes introduced in each of the other coordinates by $\delta\theta$ follow the same method. If q is any coordinate then,

$$q \rightarrow q + \delta q,$$

where $\delta q \approx (q_0' + q_1') \, \delta\theta + \tfrac{1}{2} q_0'' \, \delta\theta^2.$

Once the values of R, p_r and τ are known on the gap the effect of the RF voltage can be computed. The computation of the acceleration process uses a delta function model for the energy gain, and is identical to that used by SPRGAPZ[6]. Before proceeding to perform the transfer up to the next gap, the values of the displacements on a radial line are again required. Since the energy has changed, and thus the EO has changed, the above process must be repeated. Once this correction is made everything is set to make the transfer to the next gap.

Program Algorithms

SOMA is designed to operate as a self contained unit with the exception of the magnetic field grid which must be produced by a separate program. At the beginning of each run the program either computes the transfer matrix elements or reads them in from a binary file produced during a previous run. If the matrices are to be computed then the magnetic field and gap table are read in. Then the program searches for equilibrium orbits using the procedure of Gordon and Welton[7], for a set of specified energies. After each EO is found a search is made for the points at which the EO crosses the gaps, and when found the values of r, p_r and θ at these points are stored. A separate routine is then used to integrate the equations for the matrix elements along the EO from one gap location to the next. The integration technique is again a standard Runge-Kutta[8]. In the input stream it can be specified if only first order elements or both first order and second order elements are to be collected. The input can also specify up to 10 fixed angles at which the transfer matrix coefficients will be stored. Currently the fixed angles must fall on a standard Runge-Kutta step. After this procedure is repeated for all the selected energies, and if the main probe option is selected, then a probe transfer matrix is computed for each energy by integrating from last gap up to the track location, (which has been input as a table of r, θ values). Finally all the transfer matrices are stored on binary files.

Once SOMA has an appropriate set of transfer matrices, it then reads in the parameters common to all particles. This includes the dee voltage, the locations of slits and probes, and the set of transfer equations required. The next step is computing the starting conditions for all the particles (up to 1,000 may be run) and storing them. Particles are then run one at a time, each being run until it reaches a turn limit, an energy limit, or a radius limit, whichever comes first.

At each gap a test is performed to determine if any requested fixed angles fall between the current gap and the next gap. If a fixed angle is found then the particle parameters ($x, p_x, z, p_z, \tau, r, p_r$) are computed for that angle and stored. After all the fixed angles found have been computed, the program proceeds to compute the transfer to the next gap. At the gap, the parameters (E, x, p_x, z, p_z, τ) are updated and the process repeats itself as often as necessary. It should be noted that

the computations for the fixed angles in no way affect the values at the gap.

The fixed angles can be designated as one of two things, either a flag or a probe. The flags themselves are divided into two groups, intercepting and non-intercepting. At a flag if the particle lies between the minimum and maximum values for that flag, the orbit parameters are stored. If the flag is intercepting then the particle is considered removed from the beam, and the next particle is begun, otherwise the run continues unchanged. After all the particles have been run the program will produce scatter plots of any pairs of the orbit coordinates, at any of the possible 20 flag locations. A slit can be described as 2 intercepting flags located at the same azimuth. For detailed raytracing the particle parameters at all gaps and azimuths (or some combination thereof) can be printed out.

A probe consists of a differential and a main jaw, which can have up to 3 vertical divisions or 60 phase divisions. The probe is considered to move outward in radial steps. Upon finding the orbit parameters at the probe azimuth the program determines in which steps the particle would give a current reading. The requirements for this are that the probe location does not intercept an earlier turn of the same particle, but does intercept the current turn. For each bin that these requirements are met the bin count is augmented by 1. A possible 20,000 bins are available to be divided between the z (or φ) bins and the radial bins. At the end of the run the probe bin values are written in a binary file which can then be used as input to a plotting routine. Figure 1 shows a sample probe plot and Fig. 2 is one of the many possible scatter plots.

Comparison of SOMA with SPRGAPZ

The program SPRGAPZ integrates the exact median plane equations of motion, and the linearized z motion equations. This allows the coupling of the x motion into the z motion, but not the z motion into the x motion. There are three areas from which one expects to generate differences between SOMA and SPRGAPZ. The most obvious source is the transfer matrices themselves. As the transfer matrix technique is an approximation of a given order there will be contributions from the higher order terms. In this case it is expected that the error would be proportional to the next term in the Taylor expansion. In Fig. 3 the differences after one turn (without acceleration) are shown. If the expansion is done to first order the error function goes as $(5.6E-6)f^2$, where the initial conditions of the particle lie on the boundry of an ellipse with an emittance of $0.2f^2$ mm-mrad. At this radius, (16"), an emittance of 5 mm-mrad corresponds to a maximum orbit center displacement of 0.03". If the expansion is taken to second order the the error is proportional to $(1.6E-8)f^3$.

There is also a difference generated by the interpolation of the matrix elements when the orbit's energy lies between the stored values. For a range of different step sizes a ray was run whose energy was exactly halfway between two stored values, (the worst possible case). The initial condition of the ray was a displacement of 0.003", and the results were checked after one turn without acceleration. At 11 MeV a step size of .2 MeV results in differences of less than a tenth of a mil. Larger step sizes lead to much larger errors.

The third source of differences is the gap crossing routine. In Fig. 4 the differences after 100 turns with acceleration are shown as a function of the initial displacement from the EO. These differences are the sum of all three sources of error. As can been seen in the figure the differences for both first and second order are quadratic in the initial displacement, and at large values a cubic term appears. The fact that the error with the second order expansion is also quadratic indicates that there are residual second order effects.

Fig. 2--A scatter plot at a non-intercepting flag. In this case the radius is plotted against the particle phase. This type of output can be used to determine the correct location for a slit.
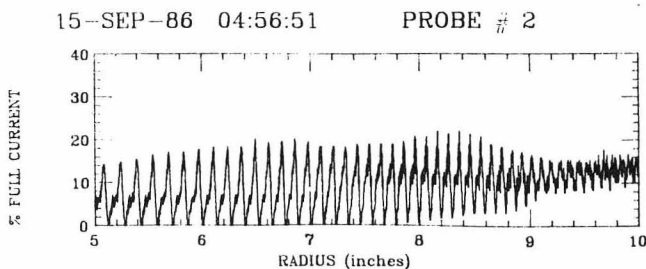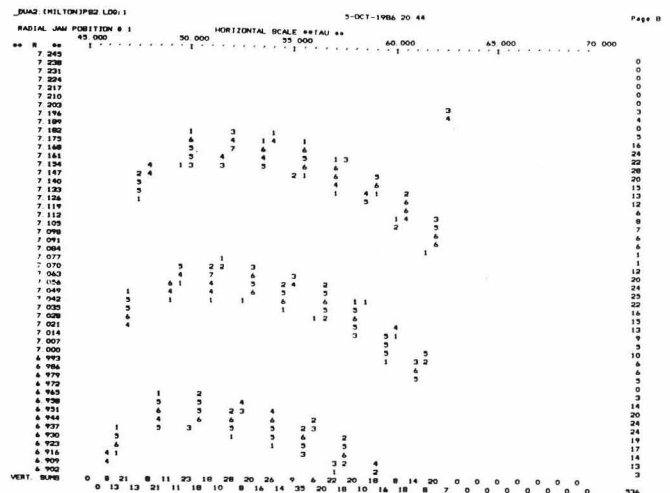

Fig. 1--A differential probe trace as computed by the program SOMA. The differential head is 0.010", and initial phase width was 14°, and the emittance was 100π mm-mrad at injection.
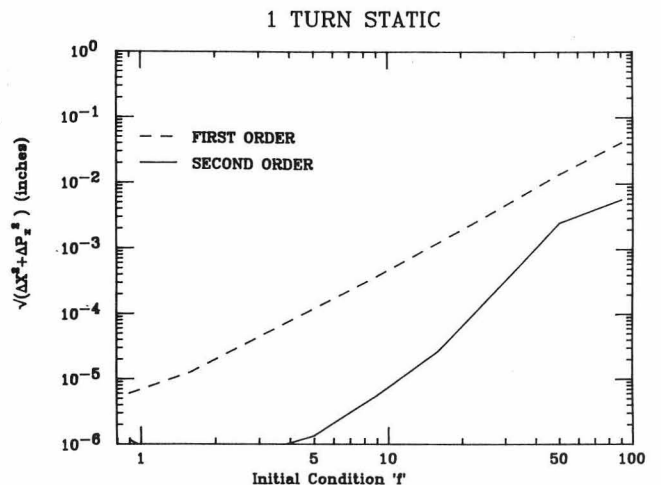

Fig. 3--The differences between SPRGAPZ and SOMA for different initial conditions. The initial conditions lie on the perimeter of an eigen-ellipse of area $(0.2)f^2$ mm-mrad. The error in the first order case is $(5.6E-6)f^2$, and in the second order case is $(1.6E-8)f^3$.

Also shown in in Fig. 4 is a comparison of a first order transfer with first order gap crossings, and a first order transfer with second order gap crossings.This illustrates that the orbits are relatively insensitive to the gap position. For the rays of small initial emittances the error in the energy is 5 parts in 10^7 (8.5 eV), which is the same magnitude as the round-off error.
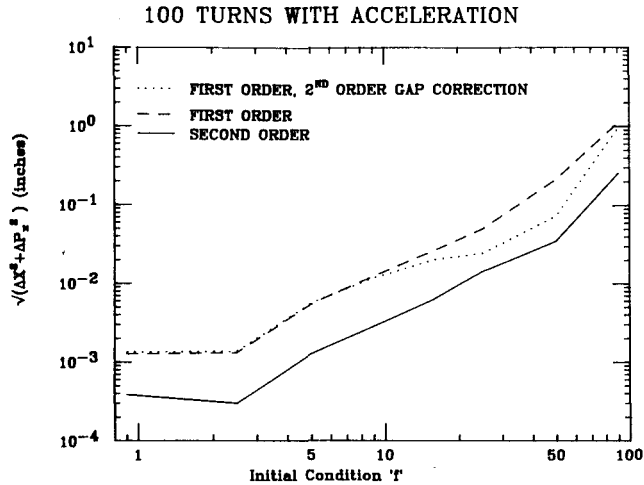
100 TURNS WITH ACCELERATION

Fig. 4--The differences between SPRGAPZ and SOMA after 100 turns with acceleration for different initial conditions. In both first and second order cases the error function is quadratic in f except beyond f=50 where the second order begins a sharp upward turn.

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