RESONANCES IN AVF CYCLOTRONS RESULTING FROM COUPLING BETWEEN MAGNETIC FIELD AND DEE-STRUCTURE

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Abstract

A general relativistic Hamiltonian is derived which describes energy, phase and centre coordinates of an accelerated particle in an AVF cyclotron. In this Hamiltonian terms are present which result from coupling between the azimuthally varying magnetic field and the acceleration structure. One example of this coupling is the well known gap crossing resonance.
We find that for certain combinations of the magnetic field symmetry number N and the number of dees a new term may be present in the Hamiltonian which affects the energy and the centre position phase of the particle. As an example this term has been given for the minicyclotron ILEC in construction at the University of Eindhoven (having N = 4, two dees for 2nd harmonic acceleration and two flattop dees in order to obtain a well defined energy in the extracted beam). The derivation of the Hamiltonian combines theories for non-accelerated and accelerated particles developed earlier in our group. The theory for non-accelerated particles was reformulated such that the acceleration process could be incorporated easily. The treatment of acceleration has been generalized such that most practical dee systems, including spiral dees, are incorperated. Furthermore, different dee systems are treated simultaneously. The final Hamiltonian contains only slowly varying terms so that the equations of motion derived from it can be integrated easily on a small computer.

1. Introduction

During the past thirty years many papers have been published which deal with the theoretical description of orbits in cyclotrons. Initially much attention was given to the time independent orbit behaviour.[1-3] In recent years progress has been made with regard to the influence of dee structures on the orbit behaviour[4,5] As a first approach acceleration may be studied in combination with a homogeneous magnetic field. This simplifies the analysis and the effects related to the specific dee structure are clearly shown. Moreover, acceleration in a homogeneous field can be analysed very nicely in a cartesian coordinate system. Afterwards the theory may be generalized by just adding the models for non-accelerated motion in an AVF cyclotron and accelerated motion in a homogeneous magnetic field together.[4] It is obvious however, that resonances arising from interaction between the magnetic field flutter and the geometrical structure of the dees are not included in such an analysis. An example of this coupling is the well known gap crossing resonance first reported by Gordon.[6] In order to describe these effects the azimuthally varying magnetic field has to be incorporated right from the beginning. An onset for such a theory has been given in ref. 7. Here expressions are derived for the gap crossing resonance in a three-sector cyclotron with one dee. Extension of this model becomes however rather complicated due to the cartesian coordinate system which was used.
In this paper we derive a general Hamiltonian theory for the accelerated motion in AVF cyclotrons. In section 2 we define the magnetic field shape and the electric potential function and give the Hamiltonian for the motion of the particle. For convenience we ignore the vertical motion. In section 3 we generalize a treatment for the time independent orbit behaviour as developed in ref. 3 in such a way that also acceleration effects can be taken into account.
Some transformations are applied to remove fast oscillating terms in the Hamiltonian. The final Hamiltonian describes the coordinates of the orbit centre and also the energy and phase of the particle. Furthermore we obtain expressions for the coordinates of the particle as function of the canonical variables. These expressions are needed for the study of the accelerated motion. In section 4 we analyse accelerated motion in a cylindrical symmetric magnetic field. We generalize an approach as given in ref. 4 such that also multy-dee systems and sprial gaps can be handled. In section 5 accelerated motion in azimuthally varying magnetic fields is studied. This results in a Hamiltonian which describes the resonances arising from coupling between the flutter and the dee structure.

2. The basic Hamiltonian

The Hamiltonian H for the accelerated motion of the particle in the median plane can be written as follows:

$$H = m_o c^2 (1+2H_{cl}/m_o c^2)^{\frac{1}{2}} + qV(r,\theta)\sin(\omega_{RF} t) \qquad (1.a)$$

$$H_{cl} = \frac{1}{2m_o}(P_r - qA_r)^2 + \frac{1}{2m_o}\left(\frac{P_\theta}{r} - qA_\theta\right)^2 \qquad (1.b)$$

where m_o is the rest mass and q the charge of the particle, c the speed of light, r and θ the polar coordinates, P_r and P_θ the canonical momenta, t the independent variable time, $V(r,\theta)$ the spacial distribution and ω_{RF} the RF-frequency of the acceleration voltage, A_r and A_θ the components of the magnetic vector potential. The vector potential is found from the magnetic field $B(r,\theta)$ in the median plane. We split $B(r,\theta)$ in an average field $\bar{B}(r)$ and a flutter profile $f(r,\theta)$, expand $f(r,\theta)$ in a Fourier series and split $\bar{B}(r)$ in a constant part B_o and a radius dependent part $\mu(r)$. This gives for $B(r,\theta)$:

$$B(r,\theta) = \bar{B}(r)\ (1 + f(r,\theta)) \qquad (2.a)$$

$$f(r,\theta) = \sum_n A_n(r)\cos n\theta + B_n(r)\sin n\theta, n=kN,\ k=1,2,3,.. \qquad (2.b)$$

$$\bar{B}(r) = B_o(1+\mu(r)),\ B_o = m_o\omega_o/q,\ \omega_o = \omega_{RF}/h \qquad (2.c)$$

with h the mode number of the acceleration and N the symmetry number of the cyclotron. Using a left-handed coordinate system, a related vector potential is given by:

$$A_\theta = -\tfrac{1}{2}B_o r(1+U) \quad ,\ U(r) = \frac{2}{r^2}\int_o^r r'\mu(r')dr' \qquad (3.a)$$

$$A_r = B_o r(1+\mu)F(r,\theta), F(r,\theta) = \sum_n \frac{A_n}{n}\sin\theta - \frac{B_n}{n}\cos n\theta \qquad (3.b)$$

We also make a Fourier analysis of the potential function $V(r,\theta)$. For convenience we only take into account the cosine components. In order to include spiral dees we assume the following distribution:

$$V(r,\theta) = \frac{\hat{V}}{2}\sum_{m=-\infty}^{\infty} a_m\cos m(\theta-\psi(r)),\ a_{-m} = a_m \qquad (4)$$

where $\hat{V}$ is the maximum dee voltage and $\psi(r)$ represents the spiraling of the dees. We insert the eqs. (3) and (4) in eq. (1) and scale the variables by deviding the momenta by qB_o, the Hamiltonian H by $m_o\omega_o^2$ and by multiplying the time with ω_o. The Hamiltonian now becomes:

$$H = \lambda^2(1+2H_{cl}/\lambda^2)^{\frac{1}{2}} + \frac{q\bar{V}}{2}\sum_{m=-\infty}^{\infty} a_m\cos m(\theta-\psi(r))\sin ht \qquad (5.a)$$

$$H_{cl} = \tfrac{1}{2}(P_r - r(1+\mu)F(r,\theta))^2 + \tfrac{1}{2}\left(\frac{P_\theta}{r} + \tfrac{1}{2}r(1+U)\right)^2 \qquad (5.b)$$

with $\lambda = c/\omega_o$ and $\bar{V} = \hat{V}/m_o\omega_o^2$

3. The time independent orbit behaviour

The Hamiltonian for the non-accelerated motion, obtained by inserting $\overline{V} = o$ in eq. (5), does not depend on t and thus is a constant of motion. Expressed in the kinetic momentum P this constant becomes: $H = \lambda^2(1+P^2/\lambda^2)^{\frac{1}{2}}$. The non-accelerated motion may now be solved by choosing $-P_\theta$ as the new Hamiltonian and θ as the independent variable. The Hamiltonian is found by solving P_θ algebraically from eq. (5). In this way the number of variables is reduced to two namely r and P_r. If only the time independent orbit behaviour has to be studied this approach works nicely.[3] For the incorperation of the acceleration process, which has to be described by four canonical variables, this method is however not convenient. We therefore derived a more general solution of the time independent orbit behaviour. Since the derivation is rather tedious we only point out the basic steps needed to obtain the final result. In order to remove the θ-dependency of the Hamiltonian we must first of all eliminate the static equilibrium orbit (SEO). The SEO is defined as a closed orbit with the same N-fold symmetry as the magnetic field. We first consider the cylindrical symmetric field for which the SEO will be a circle. Using the equations of motion derived from eq. (5) (with $\overline{V} = o$ and $F(r,\theta) = o$) we look for a solution $r = r_o$=constant, $P_r = o$. This gives a relation between the constant of motion P_θ and the radius r_o of the SEO: $P_\theta = \frac{1}{2}r_o^2(1+2\mu-U)$. Inserting this relation back in the expression for H_{cl} gives a second equation: $H_{cl} = \frac{1}{2}r_o^2(1+\mu)^2$. Comparing both equations we find that for $\mu(r) \ll 1$ the canonical variable P_θ is closely related to the energy of the particle. Therefore we change the symbols and replace in the basic Hamiltonian P_θ by E and the conjugate variable θ by ϕ and consider E as the "energy variable" and ϕ as the "phase" of the particle. Furthermore we define a radius r_o depending on E by the implicit relation:

$$E = \tfrac{1}{2}r_o^{\;2}\,(1+2\mu(r_o)-U(r_o)) \tag{6}$$

The SEO in an AVF cyclotron can now be written as:

$$R_e = r_o(1+x_e(r_o,\phi)),\quad P_e = r_o p_e(r_o,\phi) \tag{7}$$

where the yet unknown functions x_e and p_e are of the same order of magnitude as the magnetic field flutter f. The SEO can now be removed by introducting new variables $\xi = r-R_e$ and $\pi = P_r - P_e$ which describe the radial motion with respect to the SEO. In order to have a canonical transformation, also the variables E and ϕ have to be changed. We choose a generating function which depends on the old momenta P_r,E and the new coordinates $\xi,\overline{\phi}$:

$$G = -E\overline{\phi} - P_r(R_e(r_o,\overline{\phi}) + \xi) + P_e(r_o,\overline{\phi})\xi,\quad r_o = r_o(E) \tag{8}$$

The new Hamiltonian is found by expressing the old variables r, P_r, E,ϕ in the new variables $\xi,\pi,\overline{E},\overline{\phi}$ and inserting these relations in H_{cl}. It is however not possible to solve r,P_r,E and ϕ exactly from eq. (8). This difficulty can be overcome by representing H_{cl} as a power series in π and ξ:

$$H_{cl} = H_o + H_1 + H_2 + H_3 + H_4 + \,.. \tag{9}$$

where H_o is independent of ξ and π, H_1 is linear in ξ and π etc. The expansion coefficients depend on $\overline{E}$ and $\overline{\phi}$ and contain the magnetic field quantities μ,A_n and B_n. The problem may thus be approximated by taking into account terms up to first order in A_n and B_n, i.e. a first order approximation in the magnetic field flutter f. This would however be a too rough approximation. The reason for this is that in the final result to be derived, the first significant terms in H_o and H_2 are of the order f^2 and in H_3 and H_4 of the order f. Thus, in order to obtain proper results, we have to keep terms in H_o,H_1 and H_2 up to second order and terms in H_3 and H_4 up to first order in the flutter. Moreover, we have to assume that derivatives of the function μ with respect to radius are of the same order of magnitude as the flutter squared. For an isochronous magnetic field with stable vertical motion this is in fact the case. Using the approximations outlined above the relations between the old and new variables and the new Hamiltonian can be calculated. The functions x_e and p_e defined in eq. (7) are obtained from the requirement that H_1 has to vanish. In this case ξ and π describe free oscillations around the SEO. This results in a Fourier representation of the SEO. We note that if H_1 is made to vanish, the relativistic Hamiltonian H doesn't contain linear terms either. This can be verified by substituting eq. (9) in eq. (5) and expanding the square root in H. The new Hamiltonian still depends on $\overline{\phi}$ and thus contains fast oscillating terms. In order to be able to remove these terms we first introduce action angle variables I,χ which describe the radial motion in a co-moving coordinate system. The generating function for this transformation is given by:

$$G = -I\hat{\phi} - \overline{E}\hat{\phi} - I\cdot\arcsin\left(\frac{\pi}{\sqrt{2I(1+\mu)}}\right) - \frac{\pi}{2(1+\mu)}\sqrt{2I(1+\mu)-\pi^2} \tag{10}$$

with $\chi,\hat{E}$ the new momenta and I,$\hat{\phi}$ the new coordinates. Note that the function μ in eq. (10) depends on $r_o = r_o(\overline{E})$, according to eq. (6). The shape of the Hamiltonian resulting from this transformation is such that all oscillating terms can be transformed to higher order in the flutter. The method for finding the generating function is described in ref. 3. The transformation is rather complicated and we therefore only give the final Hamiltonian up to fourth degree. In each degree terms up to the first significant order are kept. Before writing down the Hamiltonian, we express the action angle variables I and χ in cartesian coordinates $x = \sqrt{2I}\cos\chi$ and $y = \sqrt{2I}\sin\chi$ which represent the position of the orbit centre (see ref. 3) and we describe the phase of the particle with respect to the co-moving coordinate system, i.e. $\tilde{\phi} = \hat{\phi} - t$ with $\tilde{\phi}$ the new phase. The final Hamiltonian H becomes:

$$H = -\tilde{E} + \lambda^2\sqrt{1+2H_{cl}/\lambda^2}$$

$$\begin{aligned} H_{cl} = \tfrac{1}{2}r_o^{\,2}(1+\mu)^2 \Bigg\{ & 1+\sum_n \frac{A_n^2+B_n^2}{2(n^2-1)} + \frac{(\nu_r-1)(x^2+y^2)}{r_o^{\,2}(1+\mu)} + \\ & + \frac{D_1(x^3-3xy^2)-D_2(y^3-3x^2y)}{24r_o^{\,3}(1+\mu)^{3/2}} + \\ & + \frac{E_o(x^2+y^2)^2+E_1(x^4-6x^2y^2+y^4)+4E_2(x^3y-y^3x)}{32r_o^{\,4}(1+\mu)^2} \Bigg\} \end{aligned} \tag{11}$$

The expressions for ν_r, D_1, D_2, E_o, E_1 and E_2 are given in ref. 3. It should be noted however that we find slightly different expressions for E_1 and E_2 due to a small error in the original derivation, namely:

$E_1 = (15A_4' + 9A_4'' + A_4''')/6$ and $E_2 = (15B_4' + 9B_4'' + B_4''')/6$
where $A_4' = (rdA_4/dr)_{r_o}$, $A_4'' = (r^2d^2A_4/dr^2)_{r_o}$, etc.

The Hamiltonian describes the position coordinates of the orbit centre x,y and the energy $\tilde{E}$ and phase $\tilde{\phi}$ of a non-accelerated particle in an azimuthally varying magnetic field. The radius r_o in the Hamiltonian has to be considered as a function of the canoncial momentum $\tilde{E}$ according to eq. (6). All field quantities in H have to be evaluated at radius $r = r_o$. The Hamiltonian does not depend on $\tilde{\phi}$ so that $\tilde{E}$ is a constant of motion as had to be expected. The isochronous magnetic field shape $B_{iso}(r)$ is found from:

$$\frac{d\tilde{\phi}}{dt} = \frac{\partial H}{\partial \tilde{E}} = \frac{\partial H}{\partial r_o}\,\frac{dr_o}{d\tilde{E}} = o \text{ with } x = y = o.$$

This leads to the expression for $B_{iso}(r)$ as derived in ref. 3.

The motion of the orbit centre is found from

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}.$$

For the study of the acceleration process we need the relations between the position coordinates r,θ and the variables $x,y,\tilde{E},\tilde{\phi}$ and t. These relations are determined by the successive transformations applied on the Hamiltonian. The relations can be written as follows:

$$r = r^{(o)} + \Delta r, \quad \theta = \theta^{(o)} + \Delta\theta \tag{12}$$

where $r^{(o)}$ and $\theta^{(o)}$ contain terms which do not depend on the Fourier coefficients A_n and B_n and Δr and $\Delta\theta$ contain all extra terms resulting from the azimuthally varying part of the magnetic field. We give $r^{(o)}$ and $\theta^{(o)}$ up to second degree in x and y and Δr and $\Delta\theta$ up to first degree and up to first order:

$$r^{(o)} = r_o\left[1+\frac{B}{r_o^2}+\frac{1}{2}\frac{A^2}{r_o^4}+\ldots\right], \quad \theta^{(o)} = \tilde{\phi}+t+\frac{A}{r_o^2}-\frac{AB}{r_o^4}+\ldots$$

$$\Delta r = r_o\sum_n\left[\frac{f_n}{n^2-1}+\frac{(n^4-4)\dot{f}_n+2(n^2-1)\dot{f}'_n}{n^2(n^2-1)(n^2-4)}\frac{A}{r_o^2}+\frac{2f_n+f'_n}{n^2-4}\frac{B}{r_o^2}\right]$$

$$\Delta\theta = \sum_n \frac{\dot{f}_n}{n^2(n^2-1)} - \frac{2(n^2-1)f_n+3f'_n}{(n^2-1)(n^2-4)}\frac{A}{r_o^2} + \frac{(n^2+2)(2\dot{f}_n+\dot{f}'_n)}{n^2(n^2-1)(n^2-4)}\frac{B}{r_o^2}+\ldots \tag{13}$$

$$A = \frac{r_o}{\sqrt{1+\mu}}\left[y\cos(\tilde{\phi}+t)-x\sin(\tilde{\phi}+t)\right]$$

$$B = \frac{r_o}{\sqrt{1+\mu}}\left[x\cos(\tilde{\phi}+t)+y\sin(\tilde{\phi}+t)\right]$$

$$f_n = A_n\cos n(\tilde{\phi}+t) + B_n\sin n(\tilde{\phi}+t)$$

$$\dot{f}_n = -nA_n\sin n(\tilde{\phi}+t) + nB_n\cos n(\tilde{\phi}+t)$$

4. Accelerated motion in a cylindrical symmetric magnetic field

We now return to the basic Hamiltonian given in eq. (5). The total Hamiltonian may be split up in two parts: $H = H_{AVF} + H_{ac}$ where H_{AVF} is the Hamiltonian for the non-accelerated motion given in eq. (11) and H_{ac} represents the effect of the RF-structure:

$$H_{ac} = \frac{q\bar{V}}{2}\sum_{m=-\infty}^{\infty} a_m\cos m(\theta-\psi(r))\sin ht \tag{14}$$

First of all the transformations derived in the previous section have to be applied on H_{ac}. This means that the relations for r and θ given in eq.(13) have to be inserted in eq. (14). The new expression for H_{ac} is found by expanding eq. (14) in a power series of the centre coordinates x and y. The result can again be splitted in two parts: $H_{ac} = H^{(o)} + \Delta H$ where $H^{(o)}$ contains the terms which are independent of the flutter and ΔH contains all extra terms arising from the azimuthally varying field. For the moment we ignore the effect of the flutter and put $\Delta r = \Delta\theta = o$ in eq. (12). We expand $H^{(o)}$ in x and y up to second degree and introduce a new phase $\bar{\bar{\phi}}$ in order to remove the spiral function ψ from the x,y-independent part of $H^{(o)}$. The generating function for this transformation is:

$$G(\tilde{E},\bar{\bar{\phi}}) = -\tilde{E}\bar{\bar{\phi}} - \int^{r_o(\tilde{E})} \frac{dE}{dr_o}\psi(r')dr' \tag{15}$$

$$\tilde{E} = \bar{\bar{E}}, \quad \tilde{\phi} = \bar{\bar{\phi}} + \psi(r_o)$$

The Hamiltonian obtained at this point still contains an oscillating linear part which has to be removed. This is done by introducing new centre coordinates which describe the centre motion of the particle with respect to the centre motion of a special solution of $H^{(o)}$ called accelerated equilibrium orbit (AEO). Of the resulting Hamiltonian we only have to keep the resonant terms. In most important order we find for $H^{(o)}$:

$$H^{(o)} = -\tfrac{1}{2}q\bar{V}\left[(\alpha_h+\alpha_{-h})\sin h\bar{\bar{\phi}} - (\beta_h-\beta_{-h})\cos h\bar{\bar{\phi}}\right]$$

$$\alpha_h = \frac{a_h}{2} + P_h\frac{D_1+\psi' D_2}{r_o} - \frac{1}{8}h^2 a_h(1+\psi'^2)\frac{D_1^2+D_2^2}{r_o^2} + \frac{1}{4}q_h\frac{D_1^2-D_2^2}{r_o^2} + \tfrac{1}{2}r_h\frac{D_1D_2}{r_o^2}$$

$$\beta_h = -P_h\frac{D_2-\psi' D_1}{r_o} + \frac{1}{8}ha_h(\psi'+\psi'')\frac{D_1^2+D_2^2}{r_o^2} + \frac{1}{4}r_h\frac{D_1^2-D_2^2}{r_o^2} - \frac{1}{2}q_h\frac{D_1D_2}{r_o^2} \tag{16}$$

$$D_1 = \frac{x\cos\psi+y\sin\psi}{\sqrt{1+\mu}}, \quad D_2 = \frac{-x\sin\psi+y\cos\psi}{\sqrt{1+\mu}}$$

$$P_h = \frac{h+1}{2}a_{h+1}, \quad q_h = \frac{h+2}{2}(h-(h+2)\psi'^2)a_{h+2},$$

$$r_h = \frac{h+2}{2}((2h+3)\psi' + \psi'')a_{h+2}$$

This Hamiltonian can be used for studying the influence of the dee-structure on the energy and centre position phase of the particle and the motion of the orbit centre. The Fourier-coefficients a_n may be obtained from measurements or numerical calculations or by assuming an idealized distribution of the acceleration voltage for which the coefficients can be calculated. For example, for a two-dee system with straight gaps (ψ=o) and half-dee anlge α:

$$a_n = \frac{2\sin n\alpha}{\pi n}(1+(-1)^n), \quad n=\pm 1, \pm 2, \ldots \tag{17}$$

where it is assumed that $V = \hat{V}$ in the dees and $V = o$ elsewhere. Substitution of this expression in eq. (16) gives the Hamiltonian derived in ref. 4. We note that the x,y-independent part of $H^{(o)}$ does not contain the spiral function $\psi(r_o)$. Therefore the energy gain per turn is independent of the spiraling of the dees.

5. Accelerated motion in an azimuthally varying magnetic field

The Hamiltonian ΔH, describing the effect of the flutter on the accelerated motion of the particle is obtained in the same way as $H^{(o)}$. We now insert the general expressions for r and θ in H_{ac}, expand the Hamiltonian in a power series of x and y and apply the transformation given in eq. (15). In the result obtained, only the resonant terms are kept. In this way we arrive at a very general expression for ΔH which can be applied for most practical dee systems. The expression is rather complicated but it simplifies considerably when one particular dee system is considered in combination with a given symmetry of the magnetic field. For convenience we only give two examples. For a 3-fold symmetric field and an "idealized" one-dee system we find up to first order in the flutter and up to first degree in x and y:

$$\Delta H = \frac{q\bar{V}}{\pi}(-1)^{\frac{h-1}{2}}\left\{-\cos h\bar{\bar{\phi}}\sum_n\frac{B_n\cos n\alpha}{n(n^2-1)} + \frac{\sin h\bar{\bar{\phi}}}{r_o\sqrt{1+\mu}}\sum_n\frac{h\sin n\alpha}{n(n^2-1)}A_n x + \frac{\cos h\bar{\bar{\phi}}}{r_o\sqrt{1+\mu}}\sum_n\frac{\sin n\alpha}{(n^2-1)(n^2-4)}\left[-x((n^2+2)B_n+3B'_n) + y\left(3nA_n+\frac{n^2+2}{n}A'_n\right)\right]\right\} \tag{18}$$

where $\alpha = \frac{\pi}{2}$ is the half dee angle.

The terms linear in x and y represent the electric gap crossing resonance first reported by Gordon [6].
The effect of this resonance is comparable with that of a first harmonic magnetic field error.
As a second example we consider the minicyclotron ILEC in construction at the Eindhoven university. This cyclotron has a 4-fold symmetry, two dees for 2nd harmonic acceleration and two flattop dees in order to obtain a well defined energy in the extracted beam.
Ignoring the effect of the flattop dees we find for ΔH:

$$\Delta H = -\frac{2q\bar{V}}{\pi}\left\{\sinh\alpha\cosh\bar{\bar{\phi}}\sum_n \frac{B_n\cos n\alpha}{n(n^2-1)} + \cosh\alpha\sinh\bar{\bar{\phi}}\sum_n \frac{A_n\sin n\alpha}{n(n^2-1)}\right\} \quad (19)$$

This expression shows that the phase $\bar{\bar{\phi}}$ of the particle changes during the acceleration when the Fourier coefficients A_n and B_n depend on radius. The phase shift can be estimated from the equation of motion for $\bar{\bar{\phi}}$:

$$\frac{d\bar{\bar{\phi}}}{d\bar{n}} = 2\pi \frac{dr_o}{d\bar{\bar{E}}}\frac{\partial \Delta H}{\partial r_o} \cong \frac{\pi}{\bar{\bar{E}}}\Delta H' \quad (20)$$

where $\bar{n}$ is the turnnumber.

For a two-dee system with half dee angle α we have in good approximation

$$\bar{\bar{E}} \cong 4q\bar{V}\bar{n}\sinh\alpha\cosh\bar{\bar{\phi}} \quad (21)$$

If we assume that $\bar{\bar{\phi}}$ is small, we may ignore the second term in ΔH which is proportional to $\sinh\bar{\bar{\phi}}$.
We then obtain for the pase shift:

$$\bar{\bar{\phi}} - \bar{\bar{\phi}}_o = -\tfrac{1}{2}\ln(\bar{n}/\bar{n}_o)\sum_n \frac{B'_n\cos n\alpha}{n(n^2-1)} \quad (22)$$

The effect is also present in a 3-sector cyclotron with three dees. In this case we find the same expression for the phase shift as given in eq. (22). For the ILEC cyclotron B_n=0 and therefore the effect is not so important. However, for a cyclotron with spiral pole tips and high mode number of acceleration h the effect may become noticeable.

Concluding remarks

The model derived in this paper has not the aim to replace numerical calculations. Numerical calculations have always to be carried out when accurate results are needed or when the electric and magnetic fields are strongly nonlinear as may be the case in the central region and the extraction region. The model can be helpful, however, as a means to check numerical calculations and as a guide for understanding special effects as well as for finding corrections for unwanted numerically or experimentally observed pertubations. Moreover, the model can be used as a powerful tool to evaluate general properties of the azimuthally varying magnetic field and the acceleration structure and to investigate the coupling effects arising from a given combination of magnetic field symmetry and geometrical structure of the dees.

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