ANALYTICAL TREATMENT OF ION INFLECTORS

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A Hamiltonian formulation is presented of particle motion in ion inflectors in cyclotron centres. The description provides the transfer through the system in a general way in terms of transfer matrices that can be used in particle and beam tracing codes like transport. Especially transparant is the treatment of the (Müller) hyperboloid inflector in which the optics is completely linear (aside from fringe field effects) allowing to calculate the transfer through the inflector without any approximation for ion beams of the proper energy and direction as well as for different ion species deviating in momentum from the inflector design value, deviating in incoming beam direction or having a large emittance. Moreover the inflector potential may be changed from the design value. Different inflector potential shapes, satisfying the Laplace equation, may be taken. As an example we have studied a potential for which the inflector surfaces only need to be bend in one dimension. The theory can be generalized to arbitrary inflector shapes in terms of expansions of the potential. This can be used e.g. for the spiral inflector, always having the electric field perpendicular to the ion velocity.

Introduction

In this paper we present a Hamiltonian derivation of the equations of motion for particles traveling in the hyperboloid (or Müller) inflector [1]). Müller invented this inflector and has proposed its use because of its superior optical properties compared to other types of inflectors, like the mirror [2,3,4]) and the spiral inflector [5]). The Müller inflector has been applied in a number of institutions like Karlsruhe, GANIL, KVI, Jülich, etc.
In order to be able to study the effect of applying different voltages to the inflector electrodes for the hyperboloid inflector, a new derivation of the equations of motion was made. In this paper we use cartesian coordinates.
The paper is organized as follows: we derive the equations of motion by introducing a generating function by which x-y-motion is decoupled. The solution is applied to the special case of the Müller inflector design values. Finally the general transfer is given. In an accompanying paper [6]) we will describe a computer program that uses the formalism of this paper, to calculate the transfer through the inflector. Useful formulas for choosing inflector and source voltage, etc. will also be given.

The Hamiltonian

In this section we write up the Hamiltonian for particles in the hyperboloid inflector field and apply a coordinate transformation in order to get decoupled motion in the three dimensions. The inflector voltage is given by:

$$V(x,y,z) = \tfrac{1}{2}k^2z^2 - \tfrac{1}{4}k^2(x^2+y^2) + V_o,$$

where k^2 determines the strength of the voltage. It is easily verified that this potential function satisfies the Laplace equation. A sketch of the collection of points in x,y,z-space, having equal voltage, is given in fig. 1. Equipotential surfaces or hyperboloids are characterized by the equation $x^2 + y^2 - 2z^2 = R^2$, with fixed R. Obviously there is rotationally symmetry about the z-axis. In for instance the z-y-plane the equipotential curves are hyperboles with asymptotes $z=\tfrac{1}{2}\sqrt{2}y$. For the hyperboloid inflector we take two equipotential surfaces with $V - V_o < o$. A particle moving in the positive z-direction enters the inflector at z=0, i.e. there are only branches of the hyperboloids with $z > 0$. Fig. 2 shows the orientation of the coordinate system as we will be using it, and a cross-section of the inflector in the z-x-plane. The magnetic field is homogenous and directed in the positive z-direction, meaning that positive particles rotate clockwise with respect to this axis. The Hamiltonian is now given by:

$$H = \frac{1}{2m}\left[(p_x+\tfrac{1}{2}eB_y)^2 + (p_y-\tfrac{1}{2}eB_x)^2 + p_z^2\right] + eV(x,y,z) \quad (1)$$

The first thing to note is that only quadratic terms appear in the Hamiltonian, implying purely linear optics. In order to get rid of annoying constants we scale the time with the cyclotron revolution frequency and introduce scaled coordinates:

$$\tilde{p}_x = p_x/eB,\ \tilde{p}_y = p_y/eB,\ \tilde{p}_z = p_z/eB$$
$$\tilde{x} = x,\ \tilde{y} = y,\ \tilde{z} = z \quad (2)$$
$$\tilde{t} = \omega_o t = (eB/m)t$$

This changes the Hamiltonian to $\tilde{H} = H/(m\omega_o^2)$. Moreover we define the dimensionless quantity $\tilde{k}^2 = ek^2/(m\omega_o^2)$. The dimension of $\tilde{p}_x$ is (m), that of $\tilde{H}$ is (m^2).
From here on we omit the wiggles above the variables, and remember that in the subsequent story we use scaled coordinates. The Hamiltonian becomes:

$$H = \tfrac{1}{2}(p_x+\tfrac{1}{2}y)^2+\tfrac{1}{2}(p_y-\tfrac{1}{2}x)^2+\tfrac{1}{2}p_z^2+\tfrac{1}{2}k^2z^2-\tfrac{1}{4}k^2(x^2+y^2) \quad (3)$$

The vertical motion is completely decoupled from the x-y-motion. However, coupling does exist between x and y motion. In order to remove this coupling in a different set of coordinates, the following generating function may be applied:

$$G=F_3(x,\bar{p}_x,y,\bar{p}_y,z,\bar{p}_z)=\bar{p}_x x+\bar{p}_y y-\bar{p}_x\bar{p}_y/(1+b)-(1+b)xy/2+\bar{p}_z z \quad (4)$$

where the quantity b is determined by the strength of the inflector field: $k^2 = -b(1+\tfrac{1}{2}b)$. Appendix 1 reveals how this generating function is found. The function G generates the following coordinate transformation: $\bar{r} = T\underline{r}$, where $\underline{r}$ is the vector with components x,p_x,y,p_y,z,p_z, and where the coordinate transformation matrix T is given by:

$$T=\begin{pmatrix} \tfrac{1}{2} & 0 & 0 & -1/(1+b) & 0 & 0 \\ 0 & 1 & (1+b)/2 & 0 & 0 & 0 \\ 0 & -1/(1+b) & \tfrac{1}{2} & 0 & 0 & 0 \\ (1+b)/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (5)$$

In the new coordinates the Hamiltonian takes the form:

$$\begin{aligned} H = {} & \tfrac{1}{2}\bar{p}_x^2(1+\tfrac{1}{2}b)/(1+b) + \tfrac{1}{2}\bar{x}^2(1+\tfrac{1}{2}b)(1+b) \\ & + \tfrac{1}{2}\bar{p}_y^2\,(\tfrac{1}{2}b)/(1+b) + \tfrac{1}{2}\bar{y}^2(\tfrac{1}{2}b)(1+b) \quad (6) \\ & + \tfrac{1}{2}\bar{p}_z^2 - \tfrac{1}{2}\bar{z}^2\,b(1+\tfrac{1}{2}b). \end{aligned}$$

Hence the solution of the inflector problem is immediately given as three separate harmonic oscillators:

$$\bar{x}'' + \omega_x^2\,\bar{x} = 0$$
$$\bar{y}'' + \omega_y^2\,\bar{y} = 0 \quad (7)$$
$$\bar{z}'' + \omega_z^2\,\bar{z} = 0$$

where the differentiation is w.r.t. the scaled time, and where $\omega_x^2 = (1+\tfrac{1}{2}b)^2$, $\omega_y^2 = (\tfrac{1}{2}b)^2$, $\omega_z^2 = -b(1+\tfrac{1}{2}b) = k^2$. (8)

For a given potential strength parameter $k^2 \leq \frac{1}{2}$ there are two values of b satisfying eq (8.c), meaning there are two different generating functions that decouple the horizontal x and y motion:

$$b = -1 \pm (1-2k^2)^{\frac{1}{2}}$$

In any case $b \leq 0$, so in actual calculations we may use the plus sign, and moreover we take

$$\omega_x = 1 + \tfrac{1}{2}b \qquad \omega_y = -\tfrac{1}{2}b \qquad \omega_z = k \tag{9}$$

in order to have positive frequencies. The eqs (7) indeed show that the optics in the inflector is completely linear.

Müller conditions

In this section we impose the condition a) that a particle entering the inflector with velocity in the positive z-direction be moving on an equipotential surface and b) that it moves in the horizontal plane after leaving the inflector. As a result a) the k^2-parameter will be fixed, as well as the velocity, and b) the dimensions of the inflector will be fixed.
The solution of eqs (7) is:

$$\begin{aligned}
\bar{x} &= x_1 \cos\omega_x t + x_2 \sin\omega_x t \\
\bar{y} &= y_1 \cos\omega_y t + y_2 \sin\omega_y t \\
\bar{z} &= z_1 \cos\omega_z t + z_2 \sin\omega_z t \\
\bar{p}_x &= \bar{x}' \,(1+b)/(1+\tfrac{1}{2}b) \\
\bar{p}_y &= \bar{y}' \,(1+b)/(\tfrac{1}{2}b) \\
\bar{p}_z &= \bar{z}'
\end{aligned} \tag{10}$$

Using transformation (5), and the frequencies (9), the solution in the old (scaled) coordinates is given by:

$$\begin{aligned}
x &= x_1\cos\omega_x t + x_2 \sin\omega_x t + y_1 \sin\omega_y t - y_2\cos\omega_y t \\
y &= y_1\cos\omega_y t + y_2 \sin\omega_y t - x_1 \sin\omega_x t + x_2\cos\omega_x t \\
z &= z_1\cos\omega_z t + z_2 \sin\omega_z t
\end{aligned} \tag{11}$$

The initial conditions are:

$$x(0) = R, x'(0) = 0, y'(0) = 0, z(0) = 0, z'(0) = v_o, \tag{12}$$

whence the solution becomes:

$$\begin{aligned}
x &= \frac{R}{1-\omega_x/\omega_y}\cos\omega_x t + \frac{R}{1-\omega_y/\omega_x}\cos\omega_y t \\
y &= \frac{-R}{1-\omega_y/\omega_x}\sin\omega_y t - \frac{R}{1-\omega_x/\omega_y}\sin\omega_x t \\
z &= \frac{v_o}{\omega_z}\sin\omega_z t
\end{aligned} \tag{13}$$

Demanding the particle trajectory to be on the hyperboloid $x^2 + y^2 - 2z^2 = R^2$ gives two conditions:

1) $\omega_x - \omega_y = 2\omega_z$

2) $\dfrac{v_o^2}{\omega_z^2} + \dfrac{2R^2}{(1-\omega_x/\omega_y)(1-\omega_y/\omega_x)} = 0$

The first condition fixes the field and the frequencies

$$k^2 = 1/6 \tag{14}$$

which also means:

$$\begin{aligned}
b &= -1 + \sqrt{2/3} \\
\omega_x &= (1 + \sqrt{2/3})/2 \\
\omega_y &= (1 - \sqrt{2/3})/2 \\
\omega_z &= (\sqrt{2/3})/2
\end{aligned} \tag{15}$$

The second condition fixes the velocity:

$$v_o^2 = R^2/24 \tag{16}$$

When we insert all this in eqs (13) we obtain the same equations for the design trajectory in the inflector as in Müller's derivation.
The condition of zero vertical velocity fixes the height of the inflector. The time at which this occurs is $\omega_z t = kt = \pi/2$, with a corresponding height $z(t)=L=R/2$.
It is easy to check that the total azimuthal θ the particle has made is $\theta = 20.23$ deg, and that the distance r w.r.t. the z-axis is $r = R\sqrt{6}/4$. Also it can easily be checked from eqs (13) that the velocity remains constant i.e. the electric field is always perpendicular to the particle velocity. Finally we note that v_o equals the radius of the orbit in the cyclotron of the particle having the corresponding velocity; $v_o = dz/dt$, with t still being the scaled time.

General transfer

The Müller conditions determine the actual design of the inflector. However the Hamiltonian (6) and the harmonic motions (7) have a more general nature. In fact they also provide the particle trajectory evolution for any initial conditions and for any (physically possible) value of b or k^2.
The solution of eq (6) or (7) can be written in matrix form:

$$\begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \\ \bar{z} \\ \bar{p}_z \end{pmatrix}_{\text{final}} = F \begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \\ \bar{z} \\ \bar{p}_z \end{pmatrix}_{\text{initial}} \tag{17}$$

with the matrix F given in table 1 and with the frequencies given in (9). The particle motion can be calculated in the following manner.
With $\underline{r}$ and $\underline{r}_f$ the initial and final vector, one obtains:

$$\begin{aligned}
\bar{\underline{r}}_i &= T\, \underline{r}_i \\
\bar{\underline{r}}_p &= F\, \bar{\underline{r}}_i \\
\underline{r}_f &= T^{-1}\, \bar{\underline{r}}_f
\end{aligned}$$

or: $$\underline{r}_f = (T^{-1} F\, T)\, \underline{r}_i \tag{18}$$

This completely describes the particle motion in the inflector. After exit from the inflector the transformation to circle and centre coordinates in the cyclotron [7] is established with the transformation T with b = 0: no electric field. Then $(\bar{x},\bar{p}_x)$ give the circle motion, or the energy and CP phase via $E = \bar{x}^2 + \bar{p}_x^2$, $\phi_{CP} = \mathrm{atan}(\bar{p}_x/\bar{x})$, and $(\bar{y},\bar{p}_y)$ give the orbit centre motion.
In this case the complete transformation is:

$$\underline{r}_{cyc} = T_{b=0}(T^{-1} F\, T)\underline{r}_i$$

TABLE 1 Coefficient of the matrix F. (17)

$$\begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \\ \bar{z} \\ \bar{p}_z \end{pmatrix}_{\text{final}} = \begin{pmatrix} \cos\omega_x t & (1+b)^{-1}\sin\omega_x t & 0 & 0 & 0 & 0 \\ -(1+b)\sin\omega_x t & \cos\omega_x t & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\omega_y t & -(1+b)\sin\omega_y t & 0 & 0 \\ 0 & 0 & (1+b)\sin\omega_y t & \cos\omega_y t & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\omega_z t & \omega_z^{-1}\sin\omega_z t \\ 0 & 0 & 0 & 0 & -\omega_z\sin\omega_z t & \cos\omega_z t \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{p}_x \\ \bar{y} \\ \bar{p}_y \\ \bar{z} \\ \bar{p}_z \end{pmatrix}_{\text{initial}}$$

Canonical and kinetic momentum

The momenta in $\underline{r}_i$ and $\underline{r}_f$ are the canonical momenta. In normal (unscaled) coordinates the relation between kinetic and canonical momentum is:

$$m\dot{x} = p_x + \tfrac{1}{2}eB_y$$
$$m\dot{y} = p_y - \tfrac{1}{2}eB_x$$
$$m\dot{z} = p_z$$

which reads in scaled coordinates:

$$x' = p_x + \tfrac{1}{2}y$$
$$y' = p_y - \tfrac{1}{2}x$$
$$z' = p_z \quad ,$$

and which can be expressed in matrix notation by a matrix M, with a 4 x 4 submatrix:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\tfrac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \tfrac{1}{2} & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}$$

When the initial and final vector are to be given in terms of kinetic momenta, the transfer through the inflector is given by a matrix S:

$$S = M^{-1}T^{-1}FTM.$$

Conclusion

The matrix method described is valied for arbitrary values of the potential strength parameter, and can be extended to other types of inflectors. In particular, for $k^2 = 1/6$ it provides the same trajectories as in Müller's approach for the hyperboloid inflector.

Appendix 1. Generating function

Start off with the function:

$$G = \bar{p}_x x + \bar{p}_y y - (1+a)\,\bar{p}_x\bar{p}_y - \frac{1}{2}(1+b)xy + \bar{p}_z z.$$

For a = b = 0 this transformation does seperate the variables in the case of only a magnetic field [7]. Here a and b are determined to do the same.

The transformation is given by:

$$x = \bar{x} + (1+a)\bar{p}_y$$
$$y = \bar{y} + (1+a)\bar{p}_x$$
$$p_x = \bar{p}_x\ \tfrac{1}{2}\ (1-a-b-ab) - \tfrac{1}{2}(1+b)\bar{y}$$
$$p_y = \bar{p}_y\ \tfrac{1}{2}\ (1-a-b-ab) - \tfrac{1}{2}(1+b)\bar{x}$$

Working out the new Hamiltonian one observes the following crossterms:

$$-\tfrac{1}{2}\,\bar{p}_x\,\bar{y}\left[b(1-\tfrac{1}{2}b-\tfrac{1}{2}ab) + k^2\ (1+a)\right]$$
$$\tfrac{1}{2}\,\bar{p}_y\,\bar{x}\left[(2+b)\ (a+\tfrac{1}{2}b+\tfrac{1}{2}ab) - k^2\ (1+a)\right]$$

These must have zero coefficients, which determines a and b:

$$1+a = (1+b)^{-1}$$
$$k^2 = -b(1+\tfrac{1}{2}b)$$

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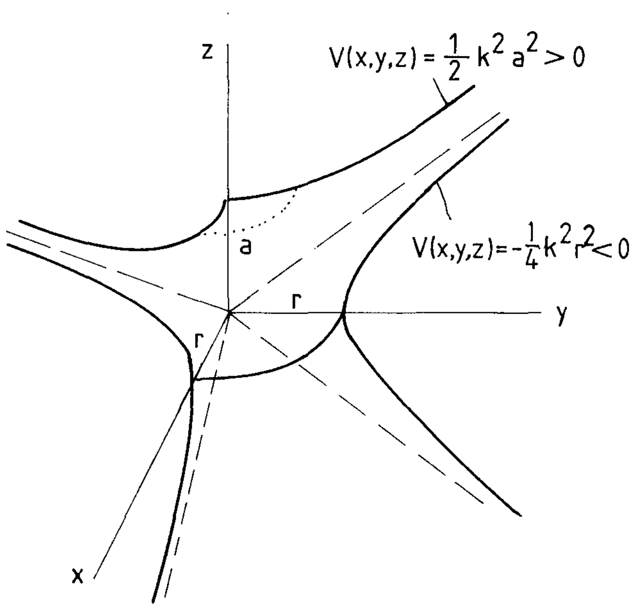

Fig. 1: Equipotential curves of the hyperboloid inflector potential

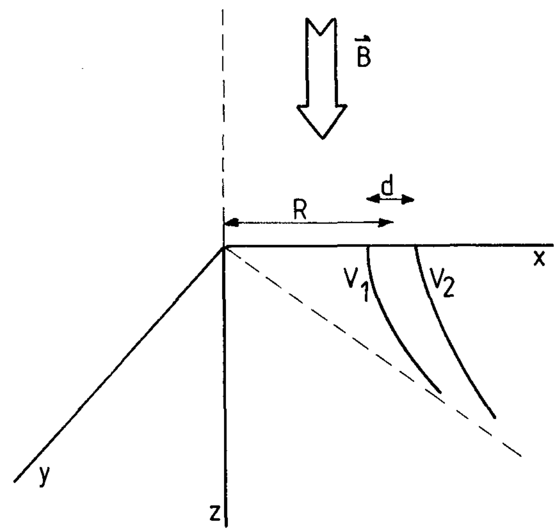

Fig. 2: Orientation of the coordinate system