CYCLOTRON MAGNETIC FIELD COMPUTATION BASED ON DATA OF AN ASYMMETRIC REFERENCE PLANE

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Summary

Analytic formulas as well as computer program are developed for cyclotron magnetic field computation based on measurement data of an asymmetric reference plane.

Introduction

It is unpractical to calculate directly the cyclotron magnetic field because of its complicated magnetic circuits. Under conventional condition, magnetic measurements only give magnetic field data on the median plane where the beam orbit is assumed to be located. However, the detailed beam dynamics requires three-dimensional magnetic field data and, hence, it is theoretically interesting and practically urgent to develop three-dimensional cyclotron magnetic field calculation based on measurement data of a reference plane. As cyclotron magnetic field calculation based on measurement data of a symmetric reference plane is well known[1], our work is devoted to cyclotron magnetic field calculation based on measurement data of an asymmetric reference plane in order to enlarge the scope of its application.

Analytic Formulas

In cylindric coordinates, let the plane Z=0 be the asymmetric reference plane where the magnetic data $Br(r,\theta,0)$, $B\theta(r,\theta,0)$ and $Bz(r,\theta,0)$ are given by magnetic measurements.

By Taylor series the three-dimensional magnetic fields read

$$Br(r,\theta,z)=\sum_{n=0}^{\infty} b_n^r(r,\theta)Z^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n Br}{\partial Z^n}\right)_{z=0}Z^n \qquad (1-1),$$

$$B\theta(r,\theta,z)=\sum_{n=0}^{\infty} b_n^\theta(r,\theta)Z^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n B\theta}{\partial Z^n}\right)_{z=0}Z^n \qquad (1-2),$$

$$Bz(r,\theta,z)=\sum_{n=0}^{\infty} b_n^z(r,\theta)Z^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n Bz}{\partial Z^n}\right)_{z=0}Z^n \qquad (1-3).$$

By Maxwell equations $\nabla\times\mathbf{H}=0$ and $\nabla\cdot\mathbf{B}=0$ with constant μ, we have $\nabla\times\mathbf{B}=0$, $\nabla^2\mathbf{B}=0$ and $\nabla^2 Bz=0$, i.e.

$$\nabla^2 Bz=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}\right)Bz=0 \qquad (2).$$

The substitution of (1-3) into (2) gives

$$\sum_{n=0}^{\infty}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_n^z(r,\theta)Z^n+\sum_{n=2}^{\infty}n(n-1)b_n^z(r,\theta)Z^{n-2}$$

$$=\sum_{n=0}^{\infty}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_n^z(r,\theta)Z^n+\sum_{n=0}^{\infty}(n+1)(n+2)b_{n+2}^z(r,\theta)Z^n=0$$

presenting the recursive formula

$$b_{n+2}^z(r,\theta)=\frac{-1}{(n+1)(n+2)}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_n^z(r,\theta) \qquad (3).$$

As $b_0^z(r,\theta)=Bz(r,\theta,0)$ from (1-3), the recursive formula (3) gives

$$b_2^z(r,\theta)=\frac{-1}{1\cdot 2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_0^z(r,\theta)$$

$$=\frac{-1}{2!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)Bz(r,\theta,0)$$

$$b_4^z(r,\theta)=\frac{-1}{3\cdot 4}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_2^z(r,\theta)$$

$$=\frac{(-1)^2}{4!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 Bz(r,\theta,0),$$

....

$$b_{2n}^z(r,\theta)=\frac{(-1)^n}{(2n)!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n Bz(r,\theta,0) \quad .(4)$$

As $b_1^z(r,\theta)=\left(\frac{\partial Bz}{\partial z}\right)_{z=0}$ from (1-3), Maxwell equation $\nabla\cdot\mathbf{B}=0$ or

$$\frac{B_r}{r}+\frac{\partial B_r}{\partial r}+\frac{1}{r}\frac{\partial B_\theta}{\partial \theta}+\frac{\partial B_z}{\partial Z}=0$$

gives

$$b_1^z(r,\theta)=-\left(\frac{B_r}{r}+\frac{\partial B_r}{\partial r}+\frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right)_{z=0}$$

$$=-\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)-\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0).$$

Consequently, by the recursive formula (3), we have

$$b_3^z(r,\theta)=\frac{-1}{2\cdot 3}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_1^z(r,\theta)$$
$$=\frac{(-1)^2}{3!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)$$
$$\times\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right],$$

$$b_5^z(r,\theta)=\frac{-1}{4\cdot 5}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)b_3^z(r,\theta)$$
$$=\frac{(-1)^3}{5!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2$$
$$\times\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right],$$

....

$$b_{2n+1}^z(r,\theta)=\frac{(-1)^{n+1}}{(2n+1)!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n$$
$$\times\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right] \quad (5)$$

Combination of (4) and (5) gives the Z component of the three-dimensional magnetic field

$$B_z(r,\theta,z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n$$
$$\times\left\{B_z(r,\theta,0)Z^{2n}-\frac{1}{2n+1}\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right]Z^{2n+1}\right\} \quad (6).$$

If the reference plane is symmetric, i.e. $B_r(r,\theta,0)=0$, $B_\theta(r,\theta,0)=0$, (6) contains only even terms, giving the well known formulas[1]

$$B_z(r,\theta,z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n B_z(r,\theta,0)Z^{2n}$$

Similarly, we get the θ and r components of the three-dimensional magnetic field:

$$B_\theta(r,\theta,z)=B_\theta(r,\theta,0)+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^{n-1}\Big\{(1-\delta_{on})$$
$$\times\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right]Z^{2n}$$
$$+\frac{1}{2n+1}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\times B_z(r,\theta,0)Z^{2n+1}\Big\} \quad (7),$$

$$B_r(r,\theta,z)=B_r(r,\theta,0)+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{\partial}{\partial r}\times\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^{n-1}$$
$$\times\Big\{(1-\delta_{on})\left[\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)B_r(r,\theta,0)+\frac{1}{r}\frac{\partial}{\partial\theta}B_\theta(r,\theta,0)\right]Z^{2n}$$
$$+\frac{1}{2n+1}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\times B_z(r,\theta,0)Z^{2n+1}\Big\} \quad (8).$$

If the reference plane is symmetric, i.e. $B_r(r,\theta,0)=0$, $B_\theta(r,\theta,0)=0$, (7) and (8) contain only odd terms, giving the well known formulas[1]

$$B_\theta(r,\theta,z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n B_z(r,\theta,0)Z^{2n+1},$$

$$B_r(r,\theta,z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^n B_z(r,\theta,0)Z^{2n+1}.$$

Implementation of the Analytic Formulas

A computer program is developed for numerical cyclotron magnetic field calculation based on measurement data of an asymmetric reference plane by the analytic formulas (6), (7) and (8). Test by a simple problem which has an exact solution shows that the analytic formulas with their computer program are good for engineering application.

Conclusion

Based on the measurement data of an asymmetric reference plane, a three-dimensional cyclotron magnetic field computation method is developed. The precision of computation by this method is considerably improved in comparison with that by the well known method[1] based on the measurement data of a symmetric reference plane. Furthermore, this method ensures far and wide application for magnetic field computation of various complicated electromagnetic devices with given measurement data of any asymmetric reference plane.

Reference

[1]. L.O.Love and W.A.Bell ORNL-3606,1964