ANALYTICAL TREATMENT OF RESONANCES

J.I.M. Botman, H.L. Hagedoorn (*) and D. Potaux (**)

(*)Eindhoven University of Technology, P.O. Box 513, 5600 MB EINDHOVEN, The Netherlands

(**) IPN, 91406 ORSAY, France

SUMMARY

Using the hamiltonian formalism the passage through resonances can be described by suitable models. In these models the most important field quantities are taken into account as radially dependent parameters. Equations are given which can be rapidly solved by a simple numerical integration routine. A few examples will be given regarding coupling resonances.

1. INTRODUCTION

Analytical models are a perfect tool to study the danger of resonances. They can predict in first order the importance of certain field quantities in a cyclotron like the harmonic components and radial derivatives in the magnetic field. A number of examples are presented in ref. 1). We will use here the same derivations but now applied to a few coupling resonances which may occur in cyclotrons. If only one single resonance is studied algebraical expressions for its importance can be made. However the problem generally becomes complicated when there is an essential radial dependency of field quantities or when there is a second resonance in its neighbourhood.

A second degree resonance will mainly suffer from second derivatives of the field quantities. A third degree resonance from third derivatives, etc. Taking lower derivatives into account in general asks for extended analytical calculations So the $v_r = 2v_z$ resonance is effected by the sum

$$\frac{1}{4}(\mu' + \mu'') + \frac{1}{16}$$

in which μ' and μ'' are the first and second radial derivatives of the mean magnetic field

$$\mu' = \frac{r}{B_o}\frac{dB_o}{dr} \qquad \mu'' = \frac{r^2}{B_o}\frac{d^2B_o}{dr^2}$$

μ'' is in general one to two orders larger than μ'. Variations in the magnetic field can occur over a characteristic length roughly equal to one half of the pole gap.This means that e.g. a third order derivative may be about $(\frac{r}{½g})^2$ larger than a first order derivative. Especially near the extraction region higher derivatives grow to large values (e.g. $\mu' \approx 0.2$ $\mu'' \approx 4$ $\mu''' \approx 80$).

As will be shown in section 2 the coupling is important if $\Delta Q \approx fA^{N-1}g^{(N)}$, where ΔQ is the distance to the resonance ($v_z + \frac{n}{m}v_x - \frac{p}{m}$; n, m, p are integers), A the amplitude of the transversal oscillation, $g^{(N)}$ the Nth derivative of a fourier component or the relative average field. The factor $f \approx 1/4$ for the $v_R - 2v_z$ resonance, $f \approx 1/8$ for the $v_R + 2v_z = 3$ resonance and $f \approx 1/32$ for the $2v_R + 2v_z = 3$ resonance. If the resonance is passed during acceleration, the value of ΔQ changes in sign. The region in which ΔQ is small must be passed in a low number of turns. As a rough criterion $2f \cdot \pi g^{(N)} \cdot A^{N-1} \cdot n \ll 1$, where n is the number of turns.

These criteria only give an indication that one must take care. Simple analytical equations of motion will show more detailed information.

In the next sections we will start from the basic hamiltonian and will keep only the most important resonant term.

2. THE BASIC HAMILTONIAN

The hamiltonian in cylindrical coordinates is given by

$$H = \frac{1}{2m}\{(\frac{p\vartheta}{r} - eA_\vartheta)^2 + (p_r - eA_r)^2 + (p_z - eA_z)^2\}$$

in which A represents the vector potential:

$$A_z = 0$$

$$A_r = -\frac{1}{2}\frac{z^2}{r}\frac{\partial B_o}{\partial\vartheta}$$

$$A_\vartheta = -\frac{1}{r}\int r\, B_o dr + \frac{1}{2} z^2 (\frac{\partial B_o}{\partial r})$$

Here B_o describes the magnetic field in the median plane (z = 0):

$$B_o = \tilde{B}(r_o)\mu = \hat{B}(r_o)\ (1 + \mu' x + \frac{1}{2}\mu'' x^2 + \Sigma(A_n + A_n' x + \frac{1}{2} A_n'' x^2 + \ldots)\cos n\vartheta + \Sigma(B_n + B_n' x + \frac{1}{2} B_n'' x^2 + \ldots)\sin n\vartheta$$

The procedure now is to take the new hamiltonian

$$K = -p_\vartheta$$

and to introduce relative variables:

$$x = \frac{r - r_o}{r_o},\quad \pi_r = \frac{p_r}{p_o},\quad \zeta = \frac{z}{r_o},\quad \pi_z = \frac{p_z}{p_o}$$

One now gets up to fourth degree in the variables.

$$K = -(1 + x)(1 - \pi_r^2)^{1/2} + \frac{1}{2}\pi_r \frac{\partial\mu}{\partial\vartheta}\zeta^2 + \frac{1}{2}\frac{(1+x)\pi_z^2}{(1-\pi_r^2)^{1/2}} + \frac{1}{8}\zeta^4 (\frac{\partial\mu}{\partial\vartheta})^2 + \frac{1}{8}\pi_z^4 + \int(1+x)\mu dx - \frac{1}{2}\zeta^2(1+x)\frac{\partial\mu}{\partial x}$$

For investigating resonances one has to keep the most important slowly changing terms. As an example for the $v_R = 2v_z$ resonance one must keep terms of the shape $x\zeta^2$, $\pi_x\zeta^2$, $\pi_x\zeta\pi_z$, $x\zeta\pi_z$, $x\pi_z^2$, $\pi_x\pi_z^2$. Using the approximation that a second derivative is much larger than a first derivative the coupling arises mainly from the seventh term in the hamiltonian:

$$-\frac{1}{2}\zeta^2 x\mu''$$

The simplified hamiltonian becomes:

$$K = \frac{1}{2}\pi_r^2 + \frac{1}{2}v_x^2 x^2 + \frac{1}{2}\pi_z^2 + \frac{1}{2}v_z^2\zeta^2 - \frac{1}{2}\zeta^2 x\mu''$$

Changing to action and angle variables:

$$x = (\frac{2J_x}{v_x})^{1/2}\cos(\varphi_x + v_x\vartheta_x) \qquad p_x = (2J_x v_x)^{1/2}\sin(\varphi_x + v_x\vartheta)$$

$$z = (\frac{2J_z}{v_z})^{1/2}\cos(\varphi_z + \frac{v_x}{2}\vartheta) \qquad p_z = (2J_z v_z)^{1/2}\sin(\varphi_z + \frac{v_x}{2}\vartheta)$$

$$K = \delta Q J_z - \frac{\sqrt{2}}{4}\mu'' \frac{J_z J_x^{1/2}}{v_z v_x^{1/2}}\cos(2\varphi_z - \varphi_x) \qquad \delta Q = v_z - \frac{v_x}{2}$$

Here the ϑ-dependency is smoothed away. This is allowed as long as the oscillating terms can be transformed to smaller terms in this order or in higher orders. This is normally the case in cyclotrons, except for the linear terms which determine the betatron frequencies. However, it is sufficient to use numerically calculated values for Q_x and Q_z, which take essentially these terms into account. The last hamiltonian is a simple one in which the quantities δQ and μ'' can be given as a function of "time" (ϑ), or with a slight change as a function of turn numer or radius [ref. 2].

One immediately observes the first criterion in the introduction: As K is a constant $\delta Q > \frac{1}{4}\mu'' A$, where A is the amplitude of the transversal oscillation and where $v_z\ v_x^{1/2} \approx 1$ is taken. The maximum growth of the resonance follows from:

$$\dot{J}_z = \frac{\sqrt{2}}{2}\mu'' \frac{J_z J_x^{1/2}}{v_z v_x^{1/2}}$$

with $J_z = \frac{1}{2} A^2$, $J_x \approx J_z \approx \frac{A^2}{2}$ one finds over n turns $\frac{\Delta A}{A} \approx n\frac{\dot{A}}{A}\cdot 2\pi \approx \frac{\pi}{2}\cdot n\cdot\mu'' A$, which yields the second criterion mentioned in the introduction: $\frac{\Delta A}{A} \ll 1$.

3. SUM RESONANCES IN A 3-FOLD SYMMETRIC FIELD

The $2\upsilon_R + 2\upsilon_z = 3$ resonance.

In a threefold symmetric magnetic field the third harmonic fourier component and its derivatives yield an intrinsic resonance. The hamiltonian with its relevant coupling term is

$$K = \frac{1}{2}\pi_r^2 + \frac{1}{2}v_r^2 x^2 + \frac{1}{2}\pi_z^2 + \frac{1}{2}v_z^2\zeta^2 - \frac{1}{4}\zeta^2 x^2 A''' \cos 3\vartheta$$

where only the cosine fourier component has been taken into account. In case of spiral sectors also the sine fourier component must be taken into account. In action and angle variables this becomes:

$$K = \delta Q J_z - \frac{1}{16}\frac{J_z J_x}{v_z v_x} A'''\cos(2\varphi_x + 2\varphi_z) \quad \delta Q = v_x + v_z - \frac{3}{2}$$

The $\upsilon_r + 2\upsilon_z = 3$ resonance.

For the $v_r + 2v_z = 3$ resonance one gets

$$K = \delta Q J_z - \frac{\sqrt{2}}{8}\frac{J_z J_x^{1/2}}{v_z v_x^{1/2}} A''_3\cos(2\varphi_z + \varphi_x)$$

$$\delta Q = (v_x + 2v_z - 3)/2$$

As an example we evaluate this latter resonance further. The equations of motion may be taken from this hamiltonian representation where in this case one may expect difficulties for $J_x \to 0$ or may be taken from a hamiltonian in cartesian coordinates

$$\sqrt{2J_x}\cos\varphi_x = x \qquad \sqrt{2J_x}\sin\varphi_x = p_x$$

$$\sqrt{2J_z}\cos\varphi_z = z \qquad \sqrt{2J_z}\sin\varphi_x = p_z$$

Then x, p_x, z, p_z have the original meanings, however slowly varying, except for a factor $\sqrt{v_{x,z}}$, which generally equals about unity.

$$K = \delta Q \cdot \frac{1}{2}(z^2 + p_z^2) - \frac{1}{v_z v_x^{1/2}}\frac{1}{16} A''_3(z^2 x - p_z^2 x - 2zp_x p_z)$$

The equations of motion following from K can be integrated e.g. with a few steps per revolution.

<u>Two resonances near each other</u>

If a coupling resonance occurs near e.g. the $v_R = 1$ resonance the danger exists that first harmonic field perturbations do increase the radial oscillation amplitudes. Then the sum resonance may be entered with too large amplitudes and the beam blows up. For investigating this combination of resonances we take slightly different action and angle variables

$$x = \left(\frac{2J_x}{v_x}\right)^{1/2}\cos(\varphi_x + \vartheta) \qquad p_x = (2J_x v_x)^{1/2}\sin(\varphi_x + \vartheta)$$

$$z = \left(\frac{2J_z}{v_z}\right)^{1/2}\cos(\varphi_z + \vartheta) \qquad p_z = (2J_z v_z)^{1/2}\sin(\varphi_z + \vartheta)$$

One finds for the hamiltonian with a first harmonic perturbation:

$$K = \delta Q_x J_x + \delta Q_z J_z - \frac{J_z}{v_z}\frac{J_x^{1/2}}{v_x^{1/2}}\frac{\sqrt{2}}{8} A''_3\cos(2\varphi_z + \varphi_x) + \sqrt{\frac{J_x}{2}}(A_1\cos\varphi_x + B_1\sin\varphi_x)$$

or in cartesian coordinates

$$K = \frac{1}{2}(A_1 x + B_1 p_x) + \frac{1}{2}\delta Q_x(x^2 + p_x^2) + \frac{1}{2}\delta Q_z(z^2 + p_z^2) - \frac{1}{v_z\sqrt{v_x}}\frac{1}{16}A''_3(z^2 x - p_z^2 x - 2zp_z p_x).$$

In fact x, p_x when multiplied with the radius yield the orbit centre coordinates. The $v_z = 1$ resonance can e.g. be excited by a second harmonic distortion of the equilibrium orbit due to second degree terms. In general this will be a small perturbation for the vertical motion and it is neglected above. The resulting equations give a stroboscopic view of the transversal motion.

4. DISCUSSION

The equations of motion can be solved by very simple integration routines. A few steps per revolution are often sufficient. As an illustration the field components belonging to the lower proton energy limit in the AGOR (ref. 3) cyclotron are used for the calculation of the passage through the $v_R + 2v_z$ resonance in the extraction region. In fig. 1 the relevant field quantities are given. The betatron frequencies have been taken from numerical calculations. In the analytical treatment the particles are started at a radius of 866 mm and accelerated through 20 revolutions. In the numerical calculations the particles are started at 867.6 mm and followed over 19 turns. The energy increase corresponds with an increase of 1 mm per turn in the radius. The results are shown in figs. 2 and 3. In fig. 4 the effect on the rotation of the phase plots caused by a slight change of the horizontal tune (dashed line in fig. 1) is shown. This small change does not influence essentially the vertical behaviour. The correspondence is quite good. The analytical calculations reveal all relevant data: dispersion, beam size, horizontal precession. Higher order effects are not taken into account. They may show distortions of the phase space figures.

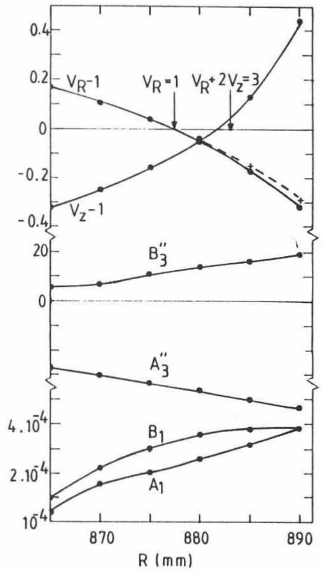

Fig.1:
The field quantities given as function of δQ_x, δQ_z, A_3'', B_3'', A and B_1 given as function of the radius. The dashed line is used for a slightly modified calculation.

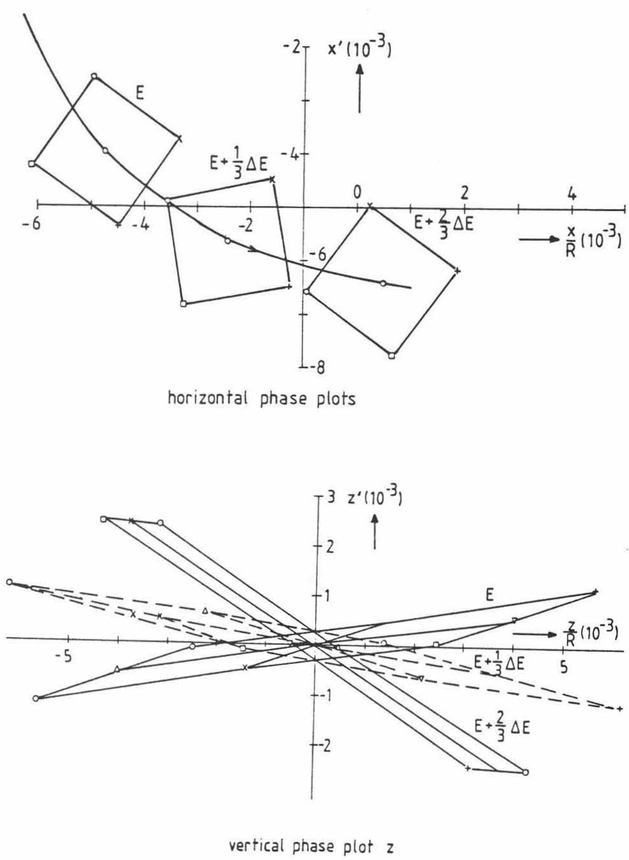

Fig.2:
Grids in phase space after 20 revolutions. The initial conditions are: for the xx' plane: $x = -10^{-3}$, $0, 10^{-3}$, $x' = -10^{-3}$, $0, 10^{-3}$, $z = 0$, $z' = 0$ and for the zz' plane $x = 0$, $x' = 0$, $z = -10^{-3}$, $0, 10^{-3}$, $z' = -10^{-3}$, $0, 10^{-3}$. The grids are started with three different initial energies: E_1, $E_1 + \frac{1}{3}\Delta E$, $E_1 + \frac{2}{3}\Delta E$, where ΔE is the energy increase per revolution.

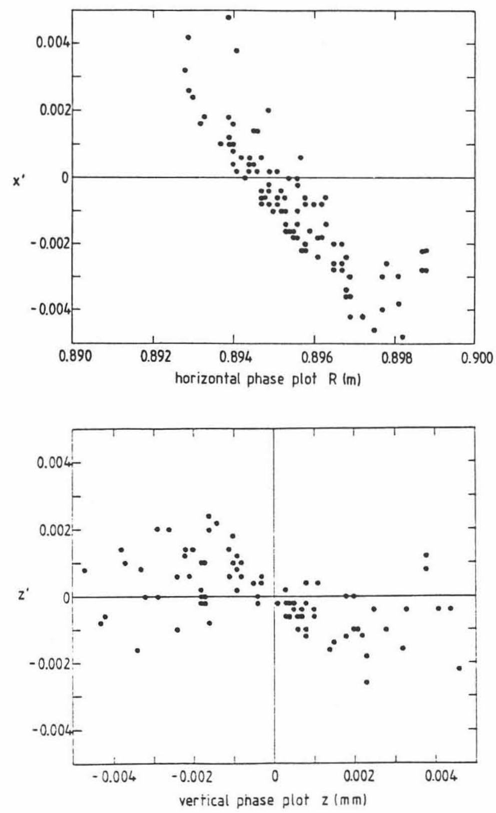

Fig.3
Numerical calculations in a real field from which the data of fig. 1 are extracted. The figures show a number of particles after 19 turns. The initial conditions are stochastically chosen corresponding to the initial regions taken for fig. 2, however, with an emittance $\epsilon = \pi * 0.27$ mmmrad. The correspondence is quite good except for a small rotation of the horizontal phase space which is ascribed to small differences in the betatron frequencies and in the energy increase per revolution. Due to the scalloping of the orbit the absolute values of the horizontal plot are shifted.

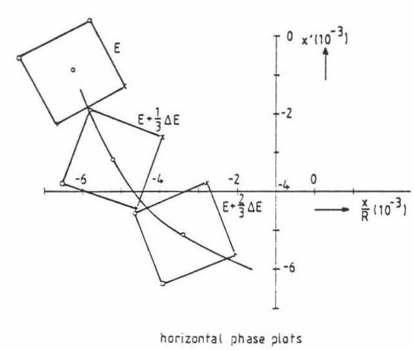

Fig.4
See caption of fig. 2. Here the data belonging to the dashed line in fig. 1 have been taken.

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