# THE INVESTIGATION OF PROTON CROSSING OF THE INTEGER RESONANCE $Q_r = 2$ IN THE CYCLOTRON BY NUMERICAL INTEGRATION OF THE NONHOMOGENEOUS LINEAR EQUACTION OF FREE RADIAL OSCILLATIONS

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ABSTRACT

The results of investigation of the proton crossing of the integer resonance $Q_r = 2$ (energy ~ 925 MeV) in a cyclotron by means of numerical integration of the nonhomogeneous linear equation of free radial oscillations are compared with the results of the analytical treatment. The damping of forced radial oscillations in the post-resonance zone is accounted for.

## Dynamic Crossing of The Integer Resonance

### 1. Numerical Simulation

The paper reports the results of investigation of the dynamic proton crossing of the integer resonances $Q_r = 2$ and 3 depending on $H_2$ and eV at $\varphi = 60^\circ$ (the injection phase $45^\circ$) in the isochronous azimuthally-symmetric magnetic field. The investigation was made by numerical integration of the nonhomogeneous linear equation of free radial oscillations

$$\rho'' + Q_r^2 \rho = -R\mathcal{E}_s \sin S\varphi. \qquad (1)$$

The energy gain per turn was realized, using four equidistant ($\varphi = 0^\circ, 90^\circ, 180^\circ, 270^\circ$) rectilinear resonators. Energy increased gradually (8 points in azimuth) inside each resonator. The main parameters of the cyclotron: $H_0 = 2$ kG, $N = 20$, $\mathcal{E}_N = 1$, $Q_z = 1.1$, $W_{in} = 663$ MeV, $1.72 \leq Q_r \leq 5$, $r_\infty = 1563.72$ cm.

In Fig.1 curves 2 and 3 describe the behaviour of the amplitude of forced radial oscillations of the proton $\rho$ and of its derivative $\rho' = \frac{d\rho}{d\varphi}$ in the process of acceleration, which are obtained by numerical integration of Eq.(1) at $H_2 = 2$ G and eV = 2 MeV/turn. Curve 5 illustrates the behaviour of the amplitude $\rho$ at $H_2 = 0.5$ G. Curve 4 describes the behaviour of the amplitude of forced radial oscillations of the proton (of the center of gravity of the beam) that was obtained by numerical integration of the nonlinear equation of motion

$$r' - \frac{2r'^2}{r} - r = -\frac{r^2}{\bar{H}R}\left[1 + \left(\frac{r'}{r_\infty}\right)^2\right]^{3/2} H_z , \qquad (2)$$

where $r' = \frac{dr}{d\varphi}$, $H_z = \bar{H} + H_N \sin\left(\frac{r}{\chi} - N\varphi\right) + H_s \sin S\varphi$, $\bar{H} = H_0\left[1 - \left(\frac{r}{r_\infty}\right)^2\right]^{-1/2}$, $S < N$, at $H_2 = 2$ G, and eV = 2 MeV/turn.

In Fig. 2 curves 1 and 2 describe the behaviour of $\rho$ and $\rho'$, obtained on the basis of the numerical integration of Eq.(1) at eV = 3 MeV/turn with the step change with radius of the second harmonic of the magnetic field $H_2$ from 5 to 0.5 G in the zone of the integer resonance $Q_r = 2$

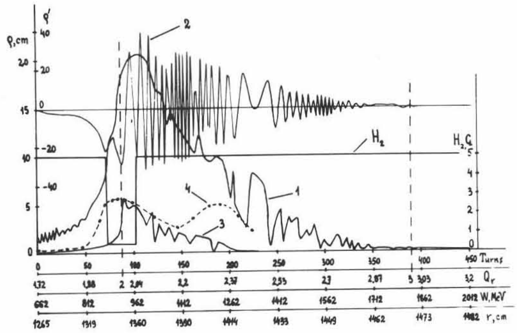

Fig. 1. The $H_2$ dependence of $\rho$ and $\rho'$ at eV = 2 MeV/turn in the proton crossing of the integer resonance $Q_r = 2$

whose length is $\Delta Q_r = \pm\ 0.05$. Curves 3 and 4 illustrate the behaviour of the amplitudes $\rho$, determined by numerical integration of Eqs.(1) and (2), respectively .

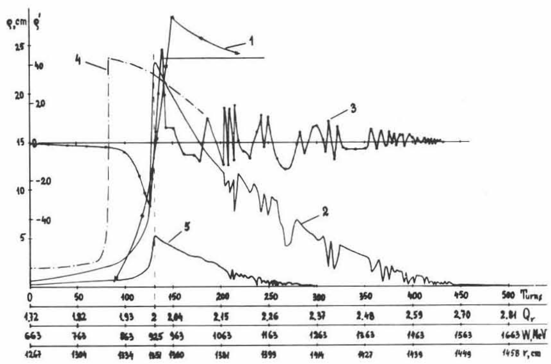

Fig. 2. The $H_2$ dependence of $\rho$ and $\rho'$ at eV = 3 MeV/turn in the proton crossing of the integer resonance $Q_r = 2$

The calculations by Eqs.(1) showed that upon reaching the maximum in the zone of the resonance $Q_r = 2$, $\rho$ and $\rho'$ decreased down to zero at $Q_r = 2.8-2.9$ and did not increase in the zone of the integer resonance $Q_r = 3$ under acceleration up to $Q_r = 3.2$.

2. Analytical Treatment

The analytical solution of Eq.(1) was traditionally described , using Fresnel's integrals [1,2]. The amplitude of forced radial oscillations of a particle can be estimated by the formula [2]

$$\rho = \frac{\pi R \varepsilon_s}{S}\left(\frac{E_0}{eV}\right)^{1/2}\left\{\frac{1}{2} \pm \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\varphi_1\sqrt{\frac{\gamma}{2}}}\cos\varphi^2 d\varphi + \frac{2}{\pi}\left[\int_0^{\varphi_1\sqrt{\frac{\gamma}{2}}}\cos\varphi^2 d\varphi\right]^2 \pm \left(\frac{2}{\pi}\right)^{1/2}\int_0^{\varphi_1\sqrt{\frac{\gamma}{2}}}\sin\varphi^2 d\varphi + \frac{2}{\pi}\left[\int_0^{\varphi_1\sqrt{\frac{\gamma}{2}}}\sin\varphi^2 d\varphi\right]^2\right\}^{1/2}, \qquad (3)$$

where the minus is for the negative values of the angle $\varphi_1$, $\gamma = \frac{dQ_r}{d\varphi} = \frac{eV}{2\pi E_0}$.

At $\varphi \to \infty$ the steady-state value of the amplitude of radial oscillation of a particle is

$$\rho = \frac{\pi R \varepsilon_s}{S}\left(\frac{E_0}{eV}\right)^{1/2}. \qquad (4)$$

This asymptotic value of the amplitude was usually regarded as the free radial oscillation amplitude of a particle . Since it is large, the radial emittance of the beam was considered to increase substantially after the passage of the interer resonance . Therefore , attempts were made to decrease the radius of a ring cyclotron and the resonance harmonic tolerance and to increase the energy gain per turn .

In Fig. 2 curve 1 illustrates the calculation of the amplitude by the formula (3) at $R = 1350$ cm, $S = 2$, $H_2 = 2$ G ($\varepsilon_2 = 5.2 \cdot 10^{-4}$), $eV = 2$ MeV/turn. The straight line is the asymptotic value of the amplitude.

The numerical integration of Eq.(2) showed that the crossing of the integer resonance $Q_r = 2$ is accompanied by the finite distortion of closed orbits with the conservation of the normalized radial emittance of a monoenergetic beam in the post-resonance zone (the phenomenon of conservation of the space-time distribution of the beam particles in the post-resonance zone [3]).

It was demonstrated in ref.[2] that the post-resonance behaviour of the amplitude of forced radial oscillations of a particle, obtained by numerical integration of Eqs.(1) and (2) and in the analytical linear consideration on the basis of the formula (3), is not adequate. Figs.1 and 2 illustrate this difference.

Thus, the formula (3) determines the maximum value of the amplitude of forced radial oscillations of a particle in the integer-resonance zone and cannot be used to determine the amplitude in the post-resonance zone when the frequency of $Q_r$ differs strongly from the integer value. I we analyze the passage of the integer resonance on the basis of the numerical integration of Eq.(1) we can conclude that it is characterized by the nonlinear properties. A reliable analytical description of the behaviour of the amplitude of forced radial oscillations of a particle in the post-resonance zone is achieved, if the solution of Eq.(1) is recorded not through Freshel's integrals but using the Bessel and Newmann functions of the index 1/4. Assuming that the frequency

$$Q_r = 1 + \frac{W_{in}}{E_0} + \frac{eV}{2\pi E_0}\varphi$$

changes linearly in azimuth, we can write in quadratures the solution to the Cauchy problem for equation (1)

$$\begin{aligned}\rho(u) = {} & \frac{\pi H_s r_\infty}{4H_0}\left(\frac{2\pi E_0}{eV}\right)^2 \int_{u_0}^{u} \frac{\sqrt{ut}}{t} \times \\ & \times\left[J_{1/4}\left(\frac{u^2}{2}\right)N_{1/4}\left(\frac{t^2}{2}\right) - J_{1/4}\left(\frac{t^2}{2}\right)N_{1/4}\left(\frac{u^2}{2}\right)\right] \times \\ & \times\sqrt{\frac{eV}{2\pi E_0}t^2 - 1}\ \sin\left[\sqrt{\frac{2\pi E_0}{eV}}\,t - \right. \\ & \left. -\left(1 + \frac{W_{in}}{E_0}\right)\frac{2\pi}{eV}\right] dt\,,\end{aligned} \tag{5}$$

where $u = \left(1 + \frac{W_{in}}{E_0}\right)\sqrt{\frac{2\pi E_0}{eV}} + \sqrt{\frac{eV}{2\pi E_0}}\,\varphi$.

Using the analytical expansion of the Bessel and Neumann functions in the case of large values of the argument $u$ and making the substitutions $t = \sqrt{\frac{2\pi E_0}{eV}}\,\beta$ and $u = \sqrt{\frac{2\pi E_0}{eV}}\,\alpha$ we get

$$\begin{aligned}\rho(\alpha) = {} & \frac{\pi H_s r_\infty E_0}{\sqrt{\alpha}\,H_0\,eV} \int_{\alpha_0}^{\alpha} \frac{\sqrt{\beta^2 - 1}}{\beta^{5/2}} \times \\ & \times \cos\frac{\pi E_0}{eV}\left[(\beta - S)^2 - \alpha^2 - 2S\alpha_0 - S^2\right] d\beta,\end{aligned} \tag{6}$$

where $\alpha_0 = 1 + \frac{W_{in}}{E_0}$.

In the stationary-phase method we get that $\rho$ is small at $\alpha < S$, at $\alpha = S$ it reaches a maximum and at $\alpha > S$ it decreases down to zero while $\alpha \to \infty$ ($\varphi \to \infty$) which agrees with the numerical integration of Eq.(1). The formula (6) describes a qualitative picture. To describe quantitatively the behaviour of the amplitude in the post-resonance zone, the dependence of the amplitude on $\alpha$ (or $\varphi$) should be stronger.

CONCLUSION

It is shown that if the term proportional to $\varrho'$ (which corresponds to friction in oscillation theory) does not occur in Eq.(1), in the numerical integration of this equation there arises the sign-inversible nonlinear radial component of velocity $\varrho' = v_r \frac{1}{\omega}$ leading to damping of the radial oscillations of a particle in the post-resonance zone. This result is supplied with an analytical explanation.

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