# THE INFLUENCE OF MAGNETIC FIELD IMPERFECTIONS ON THE BEAM QUALITY IN AN $H^-$ CYCLOTRON

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## ABSTRACT

The Hamiltonian formalism is used to derive tolerances for the magnetic field errors in an $H^-$ cyclotron in order that emittance growth due to precessional mixing or vertical amplitude growth stay below specified limits. The resonances $\nu_r = 1$, $2\nu_r = 2$, $2\nu_z = 1$, $\nu_r = 2\nu_z$ and $\nu_r + 2\nu_z = 2$ are considered. Emittance growth mainly results from the $\nu_r = 1$ and $2\nu_r = 2$ resonances. The other resonances may cause vertical beam blow up. The 30 MeV isotope production cyclotron TR30[1] is used as an example. All resonances studied have in common that $\nu_r \approx 1$ and $\nu_z \approx 0.5$ as is the case for all the important resonances in TR30.[2] The resonances are driven by errors in the vertical field. Besides that, the effect of radial field errors in the median plane on emittance is considered.

## 1. INTRODUCTION

Magnetic field errors in a cyclotron excite coherent oscillations of the beam because they displace the central orbit or they distort the transverse phase space. The total phase advance of these oscillations at extraction will differ for different turn numbers. In an $H^-$ cyclotron, because of rf phase spread, there usually are a number of different turns in the extracted beam and therefore the coherent oscillations cause an apparent increase of the emittance (precessional mixing or betatron phase mixing). This is of importance also because a larger emittance gives a larger energy spread in the extracted beam. The effect does not only depend on the magnitude of the field errors but also on the radial area in which they occur.

## 2. THE $\nu_r = 1$ RESONANCE

The $\nu_r = 1$ is a one-dimensional resonance that is driven by a first harmonic field error. The Hamiltonian describing this resonance is given by[3]

$$H = (\nu_r - 1)I + \tfrac{1}{2}\sqrt{2I}(A_1 \cos\phi + B_1 \sin\phi), \qquad (1)$$

where the azimuth $\theta$ is the independent variable, the action variable $I$ is the generalized coordinate, the angle variable $\phi$ is the generalized momentum, $A_1 = C_1 \cos\psi_1$, and $B_1 = C_1 \sin\psi_1$ are the relative first harmonic field components and $\nu_r$ is the radial oscillation frequency. In most important order the radial phase space coordinates $x, p_x$ and the orbit centre coordinates $x_c, y_c$ are related to the action-angle variables by

$$x = r_0\sqrt{2I}\cos(\phi - \theta) \quad , \quad p_x = \sqrt{2I}\sin(\phi - \theta),$$

$$x_c = r_0\sqrt{2I}\cos\phi \quad , \quad y_c = r_0\sqrt{2I}\sin\phi.$$

From Eq. (1) it follows that the first harmonic excites a coherent oscillation with an amplitude

$$\Delta x_e = \tfrac{1}{2} r_0 C_1/|\nu_r - 1|. \qquad (2)$$

The phase advance per turn of this oscillation is $\Delta\phi = 2\pi(\nu_r - 1)$, so for a turn spread at extraction of $\Delta n = 1/(\nu_r - 1)$ there is complete mixing ($\Delta\phi = 2\pi$). If the beam is initially matched and $\epsilon$ is the initial emittance, then the circulating emittance after complete mixing is (see Fig. 1)

$$\epsilon_c = \epsilon(1 + \Delta x_e/x_0)^2,$$

where $x_0$ is the initial beam size, related to the normal-

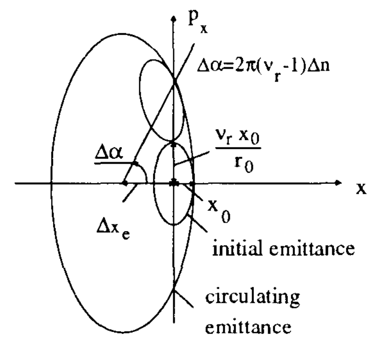


Figure 1. *Emittance growth due to a first harmonic perturbation*

ized injected emittance $\epsilon_n = \beta\gamma\epsilon$ as

$$x_0 = \left(\frac{r_0\epsilon_n}{\pi\nu_r\beta\gamma}\right)^{\frac{1}{2}} = \left(\frac{\lambda\epsilon_n}{\pi\nu_r}\right)^{\frac{1}{2}},$$

where $\lambda = m_0c/qB_0$. If the maximum allowed emittance growth factor is denoted by $f_m = (\epsilon_c/\epsilon)_{max}$ then the maximum allowable beam shift is

$$\Delta x_e \le (f_m^{\frac{1}{2}} - 1)\left(\frac{\lambda\epsilon_n}{\pi\nu_r}\right)^{\frac{1}{2}}. \qquad (3)$$

This formula together with Eq. (2) gives an upper limit for the first harmonic amplitude, assuming a coasting beam.

If the beam is not coasting but accelerated to an outer radius then the field properties may change with radius. In order to take this effect into account the orbit centre equations are derived with turn number $n$ as independent variable. These are given by

$$\frac{dx_c}{dn} = \eta y_c + \pi r_0 B_1 \ , \quad \frac{dy_c}{dn} = -\eta x_c - \pi r_0 A_1,$$

where $\eta = 2\pi(\nu_r - 1)$. Assuming smooth acceleration[4] and an isochronous magnetic field, $n$ and $r_0$ are related as follows

$$n = \alpha r_0^2, \quad \text{with} \quad \alpha = \tfrac{1}{2} q^2 \bar{B}^2/(m_0 \Delta E_{turn}), \qquad (4)$$

where $\bar{B}$ is the average magnetic field and $\Delta E_{turn}$ is the energy gain per turn. Note that $\Delta E_{turn}$ still depends on the rf phase of the particle. The orbit centre equations can be integrated for arbitrary radius dependence of $\nu_r$, $A_1$ and $B_1$. Assuming an initially centered beam ($x_c(0) = y_c(0) = 0$), the orbit centre at radius $r_0$ is

$$x_c = -2\pi\alpha \int_0^{r_0} \bar{r}^2 C_1(\bar{r}) \sin\left[\Delta\phi(r_0) - \Delta\phi(\bar{r}) - \psi_1(\bar{r})\right] d\bar{r},$$

$$y_c = -2\pi\alpha \int_0^{r_0} \bar{r}^2 C_1(\bar{r}) \cos\left[\Delta\phi(r_0) - \Delta\phi(\bar{r}) - \psi_1(\bar{r})\right] d\bar{r},$$

where $\Delta\phi(r_0)$ is the total phase advance of the betatron oscillation between $r = 0$ and $r_0$

$$\Delta\phi(r_0) = 4\pi\alpha \int_0^{r_0} \bar{r}(\nu_r(\bar{r}) - 1) d\bar{r} = \int_0^{n} \eta(\bar{n}) d\bar{n}.$$

The distance at radius $r_0$ of the orbit centre to the origin is given by

$$\begin{aligned} R \;=\; & 2\pi\alpha\Big[\left(\int_0^{r_0} \bar{r}^2 C_1(\bar{r}) \sin(\Delta\phi(\bar{r}) - \psi_1(\bar{r})) d\bar{r}\right)^2 \\ & + \left(\int_0^{r_0} \bar{r}^2 C_1(\bar{r}) \cos(\Delta\phi(\bar{r}) - \psi_1(\bar{r})) d\bar{r}\right)^2 \Big]^{\frac{1}{2}}. \quad (5) \end{aligned}$$

This represents a circle in orbit centre space which is covered with beam, if we assume complete mixing. The quantity $R$ is equivalent with $\Delta x_e$ used in Eqs. (2) and (3) but it will give a more accurate result. From Eqs. (3) and (5) it followed for TR30 that a constant first harmonic of 2 gauss would give a total emittance growth in the order of 50%.[2]

## 3. THE $2\nu_r = 2$ RESONANCE

The $2\nu_r = 2$ is a linear one-dimensional resonance that is driven by a second harmonic field error and its gradient. The Hamiltonian describing this resonance is[3]

$$H = I[(\nu_r - 1) + (\tfrac{1}{2} A_2 + \tfrac{1}{4} A_2') \cos 2\phi + (\tfrac{1}{2} B_2 + \tfrac{1}{4} B_2') \sin 2\phi],$$

where the prime denotes $rd/dr$ and with $I$ and $\phi$ defined in the previous section. It is convenient to put $B_2 \equiv 0$. The equations of motion for the orbit centre are given by

$$\frac{dx_c}{d\theta} = (C_0 - C_2) y_c, \quad \frac{dy_c}{d\theta} = -(C_0 + C_2) x_c,$$

where $C_0 = \nu_r - 1$ and $C_2 = \frac{1}{2} A_2 + \frac{1}{4} A_2'$. Thus, the frequency of the perturbed motion in orbit-centre space is given by

$$\nu_c = (C_0^2 - C_2^2)^{\frac{1}{2}}.$$

The motion becomes unstable if $|C_2| > |C_0|$. For TR30 the resonance is not crossed and therefore this instability is not expected to occur.

A more important effect is the distortion of the phase space. In the ideal case the orbit centre trajectories are circles, but with a second harmonic field error these become ellipses with aspect ratio $y_0/x_0 = ((C_0 + C_2)/(C_0 - C_2))^{\frac{1}{2}}$. To obtain a worst-case estimate of the emittance growth assume that at injection the beam is matched and there is no field error. The beam is accelerated through a region where there is a finite error and then again enters into an error free region. Assuming complete mixing the emittance growth becomes (see Fig. 2)

$$f = \frac{\epsilon_c}{\epsilon} = \frac{\pi y_0^2}{\pi x_0^2} = \frac{C_0 + C_2}{C_0 - C_2}.$$

For TR30 it followed that, for the emittance growth to be smaller than 100%, the second harmonic should be less than 50 gauss and the gradient less than 10 gauss/cm everywhere in the cyclotron.

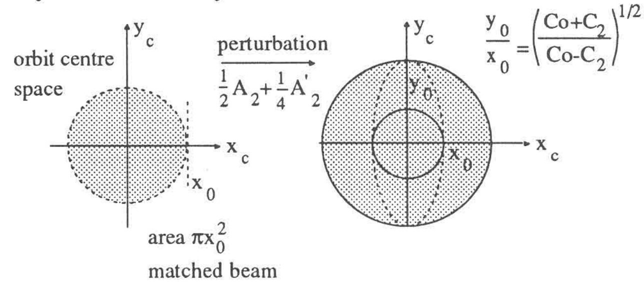


Figure 2. *Emittance growth due to a second harmonic perturbation*

## 4. THE $2\nu_z = 1$ RESONANCE

The $2\nu_z = 1$ is a linear resonance that is driven by the gradient of a first harmonic field error. Close to the median plane such a gradient produces a radial field component which is proportional to $z$. This component combines with the azimuthal velocity to give a vertical force that drives the Hill equation for $z$. The Hamiltonian for this resonance is

$$H = G[(\nu_z - \tfrac{1}{2}) - \frac{A_1'}{4\nu_z} \cos 2\psi - \frac{B_1'}{4\nu_z} \sin 2\psi]. \qquad (6)$$

The vertical phase space coordinates $z, p_z$ are related to the action angle variables $G, \psi$ as

$$z = r_0\sqrt{2G/\nu_z} \cos(\psi - \tfrac{1}{2}\theta), \quad p_z = \sqrt{2G\nu_z} \sin(\psi - \tfrac{1}{2}\theta).$$

The Hamiltonian is a constant of motion. The curves $H$=constant describe the flowlines in phase space. For

$$|\nu_z - \tfrac{1}{2}| < \frac{1}{4\nu_z}\left(A_1'^2 + B_1'^2\right)^{\frac{1}{2}},$$

the flowlines are hyperbolas and the motion is unstable. The width $\Delta r$ of the radial area that corresponds with this stopband of the resonance is given by

$$\Delta r = \left[\frac{2}{4\nu_z}\frac{(A_1'^2 + B_1'^2)^{\frac{1}{2}}}{d\nu_z/dr}\right]_{\nu_z=\frac{1}{2}}.$$

The number of turns in the resonance is $\Delta n = 2\alpha r\Delta r$ with $\alpha$ defined in Eq. (4). The amplitude of the vertical oscillation is $A_z = r_0\sqrt{2G/\nu_z}$. The amplitude growth per turn in the stopband is therefore determined by the Hamiltonian equation of motion for $G$. From this it follows that

$$\frac{1}{A_z}\frac{dA_z}{dn} = -\frac{\pi C}{2\nu_z}\sin(2\psi - \alpha_1),$$

where we introduced $A_1' = C\cos\alpha_1$, and $B_1' = C\sin\alpha_1$. For a worst-case estimate we can put $\sin(2\psi - \alpha_1) = -1$. Then

$$A_z = A_{z0}\exp(\tfrac{1}{2}\pi C\Delta n/\nu_z) \approx A_{z0}(1 + \pi C\Delta n),$$

with $\Delta n$ the number of turns in the resonance. If one allows a maximum amplitude growth $a_m = (\Delta A_z/A_{z0})_{max}$ then this determines an upper limit for $C$ which is given by

$$C < \sqrt{\frac{1}{r}\frac{d\nu_z}{dr}\frac{a_m}{2\pi\alpha}},$$

at the radius where $\nu_z = 0.5$. For TR30 the resonance is crossed at 30 cm with $\Delta r$=6.5 cm and $\Delta n$=15 turns; for an amplitude growth less than 50%, the gradient of the first harmonic at 30 cm should be less than 5 gauss/cm. At larger radii $\nu_z$ stays above but rather close to 0.5. Therefore there will again be emittance growth due to precessional mixing. Since $\nu_z \approx 0.5$, the Hamiltonian $H$ as given in Eq. (6) can be used to study this effect. This Hamiltonian has exactly the same shape as the Hamiltonian for the $2\nu_r = 2$ resonance and therefore the same analysis as given in section 3 applies.

## 5. THE $\nu_r = 2\nu_z$ RESONANCE[5,6]

The $\nu_r = 2\nu_z$ is a two-dimensional nonlinear resonance that is driven by radial derivatives of the main field. The resonance does not cause instability but it exchanges energy between the horizontal and vertical betatron oscillations. Therefore, it may become dangerous if the horizontal beam quality is bad. In that case the vertical oscillation amplitude can become large and beam can be lost at the dees. If both $\nu_r \approx 1$ and $\nu_z \approx 0.5$ then the Hamiltonian for the resonance is

$$H = (\nu_r - 1)I + (\nu_z - \tfrac{1}{2})G - \frac{g''}{\nu_z}G\sqrt{2I}\cos(\phi - 2\psi),$$

where $I, \phi$ and $G, \psi$ are the action-angle variables for the horizontal and vertical motion respectively, as defined in the previous sections. The quantity $g''$ is defined by

$$g'' = \frac{1}{4}(\bar\mu' + \bar\mu'' + \nu_z^2), \quad \bar\mu' = \frac{r}{\bar B}\frac{d\bar B}{dr}, \quad \bar\mu'' = \frac{r^2}{\bar B}\frac{d^2\bar B}{dr^2},$$

where $\bar B$ is the average magnetic field.

The quantity $2I$+$G$ is a constant of motion. To show this, introduce new variables via the canonical transformation

$$\tilde I = I, \quad \tilde\phi = \phi - 2\psi, \quad \tilde G = 2I + G, \quad \tilde\psi = \psi.$$

The new Hamiltonian is given by

$$\tilde H = \Delta\nu\tilde I + (\nu_z - \tfrac{1}{2})\tilde G - g''\frac{(\tilde G - 2\tilde I)\sqrt{2\tilde I}}{\nu_z}\cos\tilde\phi,$$

where $\Delta\nu = \nu_r - 2\nu_z$. This Hamiltonian does not depend on $\tilde\psi$ and therefore $\tilde G$ is a constant of motion

$$2I + G = \frac{1}{r_0^2}(x_0^2 + \frac{\nu_z}{2}z_0^2) = J_0 = \text{constant} > 0, \qquad (7)$$

where $x_0$ and $z_0$ are the amplitudes of the betatron oscillations. It is convenient to scale the action variable $\tilde I$ as follows

$$\rho = \frac{2\tilde I}{J_0} = \frac{(x_0/r_0)^2}{J_0}, \qquad (0 < \rho < 1).$$

Then, for a given value of $\tilde G = J_0$ the scaled Hamiltonian for $\rho$ and $\tilde\phi$ becomes

$$K = \frac{2\tilde H}{J_0} = \Delta\nu\rho - \kappa\sqrt{\rho}(1-\rho)\cos\tilde\phi, \qquad (8)$$

where the parameter $\kappa = 2g''\sqrt{J_0}/\nu_z$ is the excitation width of the resonance. It is related to the radial width of the stopband by

$$\Delta r = \frac{2\kappa}{d(\Delta\nu)/dr} = 2\kappa/(\frac{d\nu_r}{dr} - 2\frac{d\nu_z}{dr}).$$

The number of turns in the resonance is $\Delta n = 2\alpha r\Delta r$ with $\alpha$ defined in Eq. (4). The amplitude growth per turn in the stopband is determined by the Hamiltonian equation for $\rho$. From Eqs. (7) and (8) we obtain

$$\frac{dx_0}{dn} = \pi g''\frac{z_0^2}{r_0}\sin\tilde\phi, \quad \frac{dz_0}{dn} = -\frac{2\pi g''}{\nu_z}\frac{x_0 z_0}{r_0}\sin\tilde\phi.$$

The worst case estimate is obtained if the right-hand-sides of these equations are at maximum. This gives with $\nu_z = 0.5$

$$\left(\frac{dx_0}{dn}\right)_{max} = \left(\frac{dy_0}{dn}\right)_{max} = 4\pi g'' r_0 J_0.$$

From the previous analysis it is clear that the properties of the resonance depend on the beam sizes. For the initial design of TR30 the resonance would be crossed at 62.5 cm. For an estimated initial horizontal betatron amplitude of $x_0$=5 mm and a vertical amplitude $z_0$=3 mm the maximum vertical beam size after crossing the resonance (as obtained with Eq. (7)) would be $z_{max}$=10 mm, i.e. a maximum vertical blow-up with a factor three. Since the resonance was not in the fringe field, the value of $g''$ was relatively small ($g'' \approx 0.4$). This resulted in an excitation width $\kappa \approx 1.5 \times 10^{-2}$ a stopband-width of 3 cm, 13 turns in the stopband and a maximum amplitude growth per turn of 0.25 mm. Although the resonance did not seem to be too dangerous, it nevertheless was decided to avoid it by lowering the vertical tune.

## 6. THE $\nu_r+2\nu_z=2$ RESONANCE

The $\nu_r + 2\nu_z = 2$ is a two-dimensional nonlinear resonance which is driven by a second harmonic field error and its gradients. If $\nu_r \approx 1$ and $\nu_z \approx 0.5$ then the resonance is described by the Hamiltonian

$$H = (\nu_r - 1)I + (\nu_z - \tfrac{1}{2})G - a''G\sqrt{2I}\cos(2\psi + \phi - \alpha),$$

where $a''$ and $\alpha$ are defined by

$$a''\cos\alpha = \frac{A_2'' + A_2' - 2A_2}{8\nu_z}, \quad a''\sin\alpha = \frac{B_2'' + B_2' - 2B_2}{8\nu_z}.$$

The treatment is similar as for the $\nu_r = 2\nu_z$ resonance. Now, the quantity $J = 2I - G$ is a constant of motion. The resonance is a sum resonance and therefore the motion can become unstable (i.e. unbounded). For TR30 it was not considered a dangerous resonance because it is not driven by the main field but by perturbations so that $a''$ is small. Assuming the same initial beam sizes as for the $\nu_r = 2\nu_z$ resonance and $a'' < 0.1$, then is $\Delta r < 2$ cm, $\Delta n < 4$ turns and $\Delta z_0 < 0.06$ mm/turn.

## 7. RADIAL FIELD COMPONENTS IN THE MEDIAN PLANE

A radial component of the magnetic field in the median plane combines with the azimuthal velocity to give a force in the vertical direction. The k$^{th}$ cosine component $B_{r,k}$ in the Fourier expansion of the radial median plane field drives the Hill equation for $z$ as follows

$$\frac{d^2z}{d\theta^2} + \nu_z^2 z = \frac{r_0}{\bar{B}}B_{r,k}\cos k\theta, \qquad (9)$$

where $r_0$ is the radius in the cyclotron and $\bar{B}$ is the average magnetic field. The solution of Eq. (9) for a beam that initially is in the median plane ($z(0) = \dot{z}(0) = 0$) is given by

$$z(\theta) = \frac{r_0 B_{r,k}}{\bar{B}(\nu_z^2 - k^2)}(\cos k\theta - \cos\nu_z\theta). \qquad (10)$$

Thus, the radial field perturbation induces a vertical coherent oscillation. If $\nu_z$ approaches an integer ($\nu_z \approx k$), then a resonance can be excited. For TR30 this is not the case since $\nu_z \approx 0.5$ everywhere in the cyclotron. However, emittance growth due to precessional mixing once more is of importance. Since $\nu_z \approx 0.5$ only a few turns in the extracted beam will already give complete mixing. The phase advance of the $\cos k\theta$ term in Eq. (10) is equal for different turns and therefore this term can be ignored. The phase advance of the second term $\cos\nu_z\theta$ in Eq. (10) depends on turn number. The same analysis as given in section 2 for the horizontal motion applies for the vertical motion. Thus, if the maximum allowed emittance growth factor is $f_m = (\epsilon_c/\epsilon)_{max}$ then the allowable vertical beam shift is

$$\Delta z = \frac{r_0 B_{r,k}}{\bar{B}(\nu_z^2 - k^2)} < (f_m^{\frac{1}{2}} - 1)\left(\frac{\lambda\epsilon_n}{\pi\nu_z}\right)^{\frac{1}{2}}.$$

The effect of the perturbation rapidly drops with increasing $k$. For TR30 only the cases $k = 0$ (an average radial field component) and $k = 1$ (a first harmonic radial field component) are important. Allowing 50% emittance growth the average radial field value at extraction (where the effect is most pronounced) should be less than 5 gauss and the first harmonic less than 12 gauss.

## 8. REFERENCES


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