Analytical Expression of Transient Eddy Current Effect in Iron-Solid Magnets

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The transient eddy current in iron-solid magnets would disturb the magnetic field distribution at the magnet gap with variations in both space and time. In order to formalize this phenomenon by an analytical expression, we attempted to introduce the incremental permeability to Maxwell's equations together with a fringing field effect. An analytical expression for the O- and C-shape iron-core with a circular cross section is obtained by means of two-dimensional simultaneous Laplace's transformation for both the space and time. The expression is useful for estimating spatial field distribution and time constant.

1 Introduction

In a magnet with an iron-solid core the transient eddy current would be induced by changing the magnetic field and disturb the magnetic field distribution at the magnet gap with variations in both space and time. Theoretical study[1] has suggested that effect for the magnetic field due to the eddy current has time constant associated with mode of spatial magnetic field distribution. Also experimental study[2], in which transient phenomena of the magnetic field were observed for iron-solid core magnet, has been reported. It seems that such a phenomenon is due to some transient characteristics depending on the time constant associated with incremental permeability. The theoretical study, on the other hand, suggests that the eddy current effect does not depend on the permeability of the iron-core. To solve that contradiction we introduce the incremental permeability to Maxwell's equations. We also assume leakage flux ratio at the magnet gap because existence of the leakage flux would affect the time constant since larger amount of the leakage makes magnetic flux density higher in the iron-core. When discussing the eddy current, we claim that permeability for iron-core should NOT be treated as a constant value because of hysteresis. The incremental permeability can be introduced to Maxwell's equations through Faraday's law:

$$\begin{aligned} \nabla \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t} \\ &= -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{H}}\frac{\partial \boldsymbol{H}}{\partial t} \end{aligned} \tag{1}$$

In this paper, we tried to formalize the eddy current phenomena by treating the permeability as the incremental permeability $\mu_\Delta = \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{H}}$, and discuss the magnetic field distribution with variations in both space and time by assuming the ion-core whose shape was simplified properly. The leakage flux density has also been introduced to discuss that phenomena in which the time constant directly depends on the leakage flux ratio.

2 Formalization of Eddy Current Equation for O-shape Iron-core

2.1 Derivation of basic equations

Firstly, we are going to start the discussion about the eddy current with simplified iron-core model: O-shape (no gap) iron-core with length l and radius on cross section r_0, and N-turn coils are wound uniformly around the core. The Maxwell's equations considering the incremental permeability in the core are given as follows:

$$\nabla \cdot \boldsymbol{D} = 0 \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{H}}\frac{\partial \boldsymbol{H}}{\partial t} \tag{3}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{4}$$

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E}, \tag{5}$$

where we assume Ohm's law and neglect displacement current. It is adequate to take cylindrical coordinates (r, θ, z) for discussing the Maxwell's equations because of symmetry, and Faraday's law can be rewritten as follows:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rE_\theta(r,t)\right) = -\frac{\partial B_z}{\partial H_z}\frac{\partial H_z(r,t)}{\partial t}. \tag{6}$$

E_r, E_z and B_r are zero because of symmetry.

Next, we take surface integral of Ampere's law about closed curve along the core and obtain

$$lH_z(r,t) = NI(t) + l\sigma \int_r^{r_0} E_\theta(r',t)dr', \tag{7}$$

where r represents distance between symmetrical axis of the core and integrated closed curve, and $I(t)$ the coil current. By differentiating (7) with r, we obtain

$$\frac{\partial H_z(r,t)}{\partial r} = -\sigma E_\theta(r,t). \tag{8}$$

Boundary conditions for (6) and (8) are given at the surface of the core ($r = r_0$):

$$H_z(r_0, t) = \frac{NI(t)}{l} \tag{9}$$

$$2\pi r_0 N E_\theta(r_0, t) + V_0(t) = RI(t), \tag{10}$$

where R and $V_0(t)$ represent resistance of the coil and applied voltage to the coil, respectively.

We treat physical quantities at the time t as a difference from those of final states, at which the eddy current has been lost by Ohmic loss,

$$\begin{aligned} i(t) &\equiv I(t) - I_f & (11)\\ b(r,t) &\equiv B(r,t) - B_f(r) & (12)\\ h(r,t) &\equiv H(r,t) - H_f(r) & (13)\\ v(r,t) &\equiv V_0(r,t) - V_{0f}(r) & (14)\\ E(r,t) &\equiv E_\theta(r,t) - 0, & (15) \end{aligned}$$

where suffix f means the final state of each quantity, and the Maxwell's equations for these transient states are given as follows:

$$\begin{aligned} \frac{\partial h(r,t)}{\partial r} &= -\sigma E(r,t) & (16)\\ \frac{1}{r}\frac{\partial}{\partial r}(rE(r,t)) &= -\mu_\Delta \frac{\partial h(r,t)}{\partial t} & (17)\\ h(r_0,t) &= \frac{Ni(t)}{l} & (18)\\ E(r_0,t) &= \frac{1}{2\pi r_0 N}(Ri(t) - v(t)) & (19) \end{aligned}$$

where μ_Δ represents the incremental permeability.

2.2 Two-dimensional Laplace's transformations and the solutions

To solve the Maxwell's equations and the boundary conditions discussed above, we perform two-dimensional simultaneous Laplace's transformation to the equations:

$$\begin{aligned} F(q,p) &= L_{2D}[h(r,t)] & (20)\\ G(q,p) &= L_{2D}[E(r,t)]. & (21) \end{aligned}$$

After Laplace's-transforming to (16) (17) and differentiating by q, we obtain

$$\begin{aligned} F(q,p) &= \frac{C(p)}{\sqrt{q^2 - \mu_\Delta \sigma p}} & (22)\\ G(q,p) &= -\frac{qC(p)}{\sigma\sqrt{q^2 - \mu_\Delta \sigma p}} & (23) \end{aligned}$$

where $C(p)$ the function only depends on p. Now we perform inverse Laplace's transformation about q to (22) and (23),

$$\begin{aligned} f(r,p) &\equiv L_q^{-1}[F(q,p)] & \\ &= C(p)J_0(jr\sqrt{\mu_\Delta \sigma p}) & (24)\\ g(r,p) &\equiv L_q^{-1}[G(q,p)] & \\ &= \frac{C(p)}{\sigma} j\sqrt{\mu_\Delta \sigma p}J_1(jr\sqrt{\mu_\Delta \sigma p}) & (25) \end{aligned}$$

where $J_0(x), J_1(x)$ represent Bessel's functions.

From the boundary conditions we obtain

$$g(r_0, p) = \frac{1}{2\pi r_0 N}\left[\frac{Rl}{N} f(r_0, p) - w(p)\right] \tag{26}$$

where $w(p)$ represents the Laplace's transformation form of $v(t)$. We solve $f(r,p)$ and $g(r,p)$ from (24), (25) and (26):

$$\begin{aligned} f(r,p) &= \frac{Kw(p)J_0(\frac{r}{r_0}z)}{J_0(z) - \beta z J_1(z)} & (27)\\ g(r,p) &= \frac{1}{\sigma r_0}\frac{Kw(p)zJ_1(\frac{r}{r_0}z)}{J_0(z) - \beta z J_1(z)}, & (28) \end{aligned}$$

where K, β, z are defined as follows:

$$\begin{aligned} K &\equiv \frac{N}{Rl} & (29)\\ \beta &\equiv \frac{2\pi NK}{\sigma} & (30)\\ z &\equiv jr_0\sqrt{\mu_\Delta \sigma p}. & (31) \end{aligned}$$

Because the iron-core has large σ, we can neglect the second term of the denominator in (27) and (28).

Now we assume the case that the applied voltage to the coil has a step-like variation in time at $t = 0$. In that case, $v(t)$ should be treated as the delta-function because $v(t)$ is defined as the difference between two stationary states, so that $w(p)$ becomes a constant value w_0. We perform inverse Laplace's transformation about p to (27) and (28) by using Heaviside's expansion theorem and obtain:

$$h(r,t) \approx -2\frac{Kw_0}{\mu_\Delta \sigma r_0^2}\sum_n \frac{J_0(\frac{r}{r_0}z_n)}{J_1(z_n)} e^{-\frac{1}{\sigma\mu_\Delta}\left(\frac{z_n}{r_0}\right)^2 t} \tag{32}$$

$$E(r,t) \approx -2\frac{Kw_0}{\mu_\Delta \sigma^2 r_0^3}\sum_n \frac{z_n J_1\left(\frac{r}{r_0}z_n\right)}{J_1(z_n)} e^{-\frac{1}{\sigma\mu_\Delta}\left(\frac{z_n}{r_0}\right)^2 t}, \tag{33}$$

where z_n satisfies $J_0(z_n) = 0$.

From mode analysis discussed above, we conclude that the time constant of the eddy current for a n-th spatial mode is as follows:

$$\tau_n = \mu_\Delta \sigma \left(\frac{r_0}{z_n}\right)^2, \tag{34}$$

namely, the time constant of the eddy current directly depends on not only the spatial mode of the electric and magnetic field in the iron-core but also the incremental permeability of the core.

3 Formalization of Eddy Current Equation for C-shaped Iron-core

Next, we discuss C-shape iron-core magnet whose size and material are identical to O-shape iron-core discussed in the previous section except that the C-shape core has a gap whose height is g. Such an electromagnet would generate a magnetic field not only in the gap ($r \leq r_0$) but also around the gap ($r \geq r_0$) because of leakage flux. It is difficult to deal with the leakage generally since it depends on both pole shape and properties of the core. Here, we adopt an assumption in which magnetic flux density in the iron-core $B_c(r,t)$ and in the gap $B_a(r \leq r_0, t)$ are proportional to each other, namely,

$$B_c(r,t) = kB_a(r \leq r_0, t), \tag{35}$$

which represents leakage flux ratio and has a value of $k \geq 1$. This assumption violates the conservation of the magnetic flux density at the surface of the pole, however, we treat the leakage flux as a simplified parameterization given by k. Also this means that when we take surface integral of Ampere's law around closed curve along the core we only need to consider the magnetic field in the gap although it depends on the leakage flux.

Assuming (35) and considering the transient state similar to the previous section, we obtain following equations:

$$l\frac{\partial h_c(r,t)}{\partial r} + \frac{g}{k\mu_0}\frac{\partial b_c(r,t)}{\partial r} = -l\sigma E(r,t) \tag{36}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE(r,t)) = -\mu_\Delta \frac{\partial h_c(r,t)}{\partial r} \tag{37}$$

$$lh_c(r_0,t) + \frac{g}{k\mu_0}b_c(r_0,t) = Ni(t) \tag{38}$$

$$E(r_0,t) = \frac{1}{2\pi r_0 N}(Ri(t) - v(t)). \tag{39}$$

Now consider infinitesimal variation in $B_{c,z}(r,t)$ and $H_{c,z}(r,t)$ and the incremental permeability μ_Δ be kept a constant value during the change, namely,

$$b_c(r,t) = \mu_\Delta h_c(r,t). \tag{40}$$

The Maxwell's equations and the boundary conditions are then given as follows:

$$\left(l + \frac{g\mu_\Delta}{k\mu_0}\right)\frac{\partial h_c(r,t)}{\partial r} = -l\sigma E(r,t) \tag{41}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE(r,t)) = \mu_\Delta \frac{\partial h_c(r,t)}{\partial t} \tag{42}$$

$$\left(l + \frac{g\mu_\Delta}{k\mu_0}\right)h_c(r_0,t) = Ni(r,t) \tag{43}$$

$$E(r_0,t) = \frac{1}{2\pi r_0 N}(Ri(t) - v(t)), \tag{44}$$

and are resolved to that of the O-shape iron core. They can be solved by operations similar to that in case of the O-shape iron-core, and the time constant of the eddy current is given as follows,

$$\tau_n^{gap} = \frac{l\sigma}{\frac{l}{\mu_\Delta} + \frac{g}{k\mu_0}}\left(\frac{r_0}{z_n}\right)^2. \tag{45}$$

In this case, the time constant depends on both the incremental permeability and the leakage flux ratio.

4 Discussion and Summary

4.1 Behavior of the time constant

Now we discuss the time constant (45) in the case that 1)the iron-core has large incremental permeability and 2)the magnet has large amount of the leakage flux ratio.

For large μ_Δ the time constant is written as follows:

$$\tau_n^{gap} \approx k\frac{l}{g}\mu_0\sigma\left(\frac{r_0}{z_n}\right)^2. \tag{46}$$

Namely, (46) indicates that the leakage flux makes the time constant longer and it seems that k makes the gap height shorter. The time constant(46) corresponds to that of the theoretical study[1] when the leakage flux ratio is chosen to be unity, namely, $k = 1$. In this case the time constant does not depend on the incremental permeability, but, on the other hand, we need to understand some relationships between the leakage flux ratio k and the incremental permeability μ_Δ.

(45) shows another interesting behavior about the leakage flux ratio. If k has large value, so that $l/\mu_\Delta \gg g/k\mu_0$ is satisfied, the time constant would be rewritten as follows:

$$\tau_n^{gap} \approx \mu_\Delta\sigma\left(\frac{r_0}{z_n}\right)^2, \tag{47}$$

that corresponds to the result of O-shape iron core. This means that larger leakage flux ratio makes the gap height shorter effectively, so that the time constant becomes the same that for without gap. Such a case may be realized at near the edge of the pole face, and consequently there may be some dependence of the time constant on the radial position in the gap.

4.2 Suggestion for experiment

Some experiments are needed for verification of the eddy current effect. The time response of the magnetic field for an excitation voltage pattern of the coil should be observed with both iron-solid and lamination core magnet which have the same dimension each other.

According to (32) and (33), which are the solutions for the step-like variation of the applied voltage to the coil, the spatial distribution of the field tends to become larger at the edge of the pole than at the center. The time constant, on the other hand, implies that the edge of the pole tends to have longer time constant than the center. Such a dependence of the time constant on the position in the gap should also be observed, and similar experiments with some patterns of the excitation voltage to the coil are needed.

4.3 Summary

We tried to formalize the eddy current phenomena in the electromagnet and an analytical expression for simplified iron-core was obtained by means of two-dimensional simultaneous Laplace's transformation for both space and time and by introducing the incremental permeability into the Maxwell's equations.

The time constant from the analytical solution with the leakage flux ratio depends on both the spatial mode of the magnetic field and the incremental permeability, and the larger leakage flux ratio makes the gap height shorter effectively. This effect implies that there is dependence of the time constant on the position in the gap.

Both the analytical formalization and the experimental verification of the eddy current effect for the magnetic field would be able to contribute not only accelerator control but also experiments in which one uses the electromagnet. We are now preparing the experiment for the study of the eddy current phenomena in the electromagnet.

References

[1] P. J. Bryant, *CERN Accelerator School*, CERN 92-05, 52 (1992).

[2] H. N. Jungwirth, *Proc. of 9th Int. Conf. on Magnet Technology*, 138 (1985).