THEORETICAL STUDY OF EMITTANCE TRANSFER*

H. Okamoto, K. Kaneta,
Hiroshima University, 1-3-1 Kagamiyama, Higashi-Hiroshima 739-8530, Japan

A. M. Sessler,
LBNL, 1 Cyclotron Road, Berkeley, CA94720, U.S.A.

Abstract

The beam emittance, i.e. the volume of a charged-particle ensemble in six dimensional phase space, is approximately conserved in common accelerators. The concept of "cooling" is thus crucial in improving the beam quality for various machine users. In contrast to cooling, it is rather easy to change the ratios of emittance projections on to the three spatial degrees of freedom. This paper addresses a possible method of controlling the projected emittances with conservative interactions; not only a full emittance exchange but also a partial emittance transfer can readily be achieved in a dedicated storage ring operating near resonance. In a process of eimttance transfer, strong correlations between three directions are naturally developed, which may be useful for specific purposes.

INTRODUCTION

Liouville's theorem implies that the six-dimensional (6D) phase-space volume occupied by a charged-particle beam is an approximate invariant unless the beam is subjected to dissipative interactions (such as in cooling). Symplectic conditions, in a Hamiltonian system (once again, no dissipation), put constraints upon emittance transfer between the various degrees of freedom [1]. We can, however, even in non-dissipative Hamiltonian systems arrange for partial emittance transfers. This process results in phase space correlations and change in the emittance projections on to various phase planes; namely, the projected emittances in three degrees of freedom are controllable while the direction and amount of a possible emittance flow are not very flexible because of the symplectic nature of Hamiltonian systems. In some applications, it is clearly advantageous to optimize the ratios of projected emittances despite the effect of correlations. Since the three emittances are not always equally important, we may consider reducing the emittance of one direction at the sacrifice of the other emittance(s). Emittance exchanging systems have been seriously discussed these days to improve the performance of free electron lasers (FELs) [2,3].

As a possible scheme to achieve efficient emittance control, we study a compact storage ring operating near resonance. The basic features of linear and nonlinear emittance flow are briefly described with numerical examples. A general discussion touching on some of these matters was made over ten years ago and recently published in Ref. [4].

COUPLING STORAGE RING

For a full emittance exchange between the longitudinal and transverse directions, Cornacchia and Emma designed a beam transport channel that employs a special radio-frequency (rf) cavity placed in the middle of a magnetic chicane [5]. Their rectangular rf cavity, excited in a deflective mode, is identical to the *coupling cavity* previously considered for three-dimensional (3D) laser cooling [6]. Although the present scheme is based on a compact ring rather than a short beam transport, we have much higher flexibility in manipulating phase spaces. It is actually straightforward to accomplish a wide range of emittance ratios simply by switching coupling potentials on and off. Furthermore, various linear and nonlinear correlations can be introduced in phase spaces if we switch off coupling (or extract the beam) on the way to a full emittance exchange.

Neglecting interparticle Coulomb interactions, the dynamic motion of a particle in a storage ring can be approximated as the superposition of three harmonic oscillators. The Hamiltonian of interest to us is given by

$$H = \frac{1}{2}\sum_{q=x,y,z}\left[p_q^2 + \left(\frac{\nu_q}{R}\right)^2 q^2\right] + \phi_c(x,y,z;s), \qquad (1)$$

where x, y, and z stand, respectively, for the horizontal, vertical, and longitudinal directions, ν_q $(q = x, y, z)$ is the tune of each direction, R is the average radius of the ring, and the independent variable s is the path length measured along the design beam orbit. Here, $\phi_c(x,y,z;s)$ is an artificial potential that couples three degrees of freedom if required. As an example, let us take a symmetric coupling

$$\phi_c / R = g_x x^m z^n \delta_p(s - s_x) + g_y y^m z^n \delta_p(s - s_y), \qquad (2)$$

where m and n are positive integers, and $g_{x(y)}$ are coupling constants. Since coupling sources are generally electromagnetic devices localized and fixed at specific positions of the ring, we have multiplied each coupling term by a periodic delta function $\delta_p(s)$ with periodicity $2\pi R$. For the sake of simplicity, we assume that the two

*Work supported in part by Japan Society for the Promotion of Science, and by the U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. DE-AC02-05CH11231.

coupling sources are sitting side by side; namely, $s_x \approx s_y$. Without loss of generality, we set the origin of the *s*-coordinate at the position of the coupling potential.

The coupling itself can be strengthened by increasing $g_{x(y)}$, but that is not practical, nor necessary. We can prove that, in order to expedite an emittance flow from one direction to another, the two directions have to be on coupling resonance. Typically, we employ difference resonance:

$$m\nu_x - n\nu_z \approx \ell_x, \tag{3a}$$

$$m\nu_y - n\nu_z \approx \ell_y, \tag{3b}$$

where $\ell_{x(y)}$ are integers. The efficiency of the emittance transfer can be controlled by changing the distance from the resonance or the coupling constant. Although most accelerator designers try to avoid such resonances, we rather exploit them here. It is straightforward to derive the following invariant under the conditions (3):

$$I \equiv \frac{\varepsilon_x + \varepsilon_y}{m} + \frac{\varepsilon_z}{n} = \text{const.}, \tag{4}$$

where ε_q $(q = x, y, z)$ is the projected emittance in *q*-direction. Due to the existence of this invariant, the sum of all three projected emittances is always bounded. Figure 1 shows an example of typical emittance flow when a linear coupling potential ($m = n = 1$) is taken into account. Clearly, the sum of the three projected emittances is conserved; namely, $\varepsilon_x + \varepsilon_y + \varepsilon_z = \text{const.}$ Needless to say, we can stop the emittance flow at any timing by shutting down the coupling sources, which allows us to choose various ratios of projected emittances.

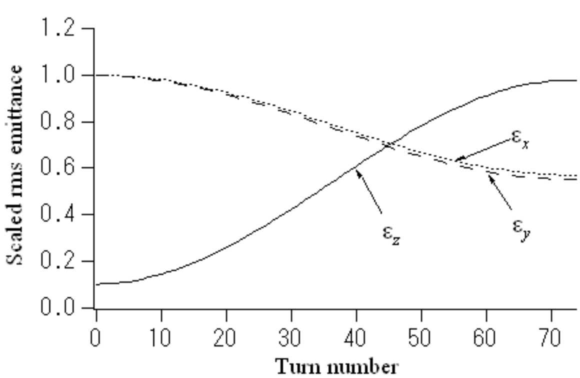

Figure 1: Time evolution of root-mean-squared (rms) projected emittances in the *x*, *y*, and *z* directions. A linear coupling ($m = n = 1$) with $g_x = g_y = 0.01$ and $R = 1$ has been assumed in this example. The tunes are $(\nu_x, \nu_y, \nu_z) = (1.1, 1.1, 0.1)$ satisfying the resonance conditions in Eqs. (3). The initial distributions of particles are Gaussian and upright in all phase planes. At injection, the emittance ratio is $\varepsilon_{x(y)} / \varepsilon_z = 10$.

There is a particularly simple scaling law in the case of symmetric coupling where $g_x = g_y (\equiv g)$ and $\nu_x = \nu_y$ $(\equiv \nu_\perp)$. When the ring is exactly on resonance, the emittance-evolution pattern itself is basically independent of the fundamental parameters (g, $\nu_\perp$, and R) and only the exchange period varies. We can, therefore, carry out a desired beam optimization sooner or later even if it is difficult to increase the coupling strength in practice. An identical emittance-transfer process can be expected under the condition $g(mR/\nu_\perp)^{m/2}(nR/\nu_z)^{n/2} = \text{const.}$

PHASE-SPACE CORRELATIONS

Linear Correlation

In a linear situation as shown in Fig. 1, the direction of a possible emittance flow is strictly limited due to symplecticity [1]; provided the beam is initially matched to the uncoupled system where $\phi_c = 0$, the projected emittances only start to flow from a "hot" to a "cold" degree of freedom. We, however, notice that the emittance flow is eventually reversed after the three emittances become equal. This suggests the development of correlation among the degrees of freedom during the emittance exchanging process, which is confirmed in Fig. 2. The correlation disappears when a full emittance transfer is accomplished.

The projected emittance of a particular direction can be diminished if we could somehow produce a beam with such a correlation. It is, however, impossible to introduce the necessary correlation into phase space (with conservative forces) while preserving the three projected emittances; the sum of the three projected emittances inevitably grows. As a result, the minimum emittance of any direction will be larger than the level achievable with the original uncorrelated distribution.

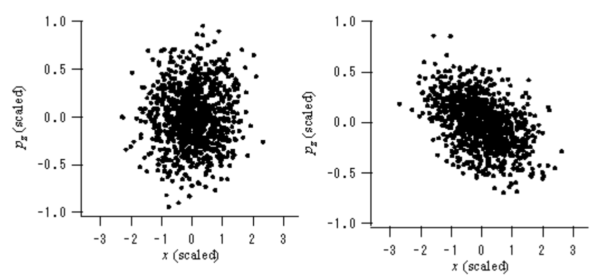

Figure 2: Phase-space correlation developed in the coupled harmonic-oscillator system under a linear difference resonance. The particle distributions at injection (left) and at the 40th turn (right) in Fig. 1 are plotted on a correlation phase plane spanned by *x* and p_z.

Nonlinear Correlation

When the coupling source is nonlinear, the direction of emittance flow can be made more flexible. The quality of a beam is, in a practical sense, somewhat deteriorated by "smearing" in phase space. (The 6D volume of the beam is, of course, unchanged as long as the nonlinear force is conservative.) The coupling constant has to be moderate to avoid too much a shrinkage of the dynamic aperture. The direction and amount of a nonlinear emittance flow depend basically on initial beam conditions (the three projected emittances) as well as lattice parameters (tunes and coupling constants). Unlike linear coupling cases where particle distributions in 2D phase space are always elliptic as shown in Fig. 2, complex correlations can be developed with nonlinear potentials because of the amplitude dependence of single-particle tunes. An example of nonlinear correlation is given in Fig. 3 where third-order coupling ($m=1$, $n=2$) has been assumed. To be on resonance, the tunes have been set at $(\nu_x, \nu_y, \nu_z) = (1.2, 1.2, 0.1)$. These distributions in p_x-p_z and z-p_x phase planes correspond to what we observe at the 120th turn after injection. The correlation tends to be less pronounced due to nonlinear smearing as the beam stays in the coupling ring for a longer period.

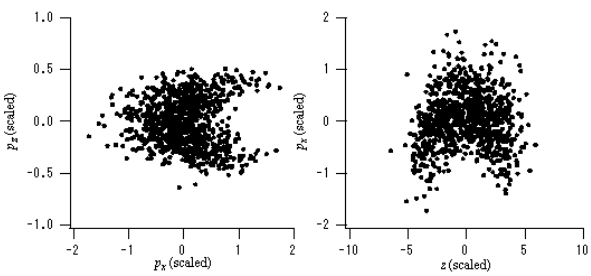

Figure 3: Phase-space correlation developed in the coupled harmonic-oscillator system under a nonlinear (third-order) difference resonance. The coupling constants and average ring radius have been chosen to be $g_x = g_y = 1$ and $R = 1$. The initial emittance ratio assumed here is $\varepsilon_{x(y)} / \varepsilon_z = 0.1$.

COUPLING SOURCES

Transverse-transverse Coupling

In order to introduce linear or nonlinear coupling between the two transverse directions, we can simply employ multipole magnets. For linear coupling, a skew quadrupole magnet can be used. (A solenoid is another good candidate as a linear coupling source.) Multipole magnetic fields are derivable from vector potentials that generally have only longitudinal components. For instance, the potential of a skew quadrupole is given by

$$\mathbf{A}_{skew} = \left(0, \quad 0, \quad g_{skew} xy \delta_p(s)\right), \tag{5}$$

where g_{skew} is proportional to the strength of the magnetic field. For nonlinear coupling, we excite a sextupole (third order) or an octupole (fourth order) magnet.

Transverse-longitudinal Coupling

Special rf cavities can be utilized to correlate the longitudinal motion with the transverse. The coupling cavity mentioned above is a simple solution for generating linear synchro-betatron coupling [6]. Even a regular accelerating cavity can be a linear coupling source when it is placed at a position with finite momentum dispersion [7]. The rectangular coupling cavity operating in TM_{210} mode produces a longitudinal electric field proportional to x, which establishes a linear correlation between the horizontal and longitudinal directions. To make a direct coupling with the vertical direction, we just rotate the cavity by 90 degrees around its axis of symmetry.

As to nonlinear coupling, we may exploit a cylindrical cavity rather than a rectangular type. The electromagnetic fields of a cylindrical resonator excited in $TM_{k\ell 0}$ mode can again be obtained from a vector potential with no transverse components:

$$\mathbf{A}_{rf} = \left(0, \quad 0, \quad \frac{V}{\omega} J_k(\zeta_{k\ell} r / r_0) \cos(m\theta) \sin(\omega t) \delta_p(s)\right), \tag{6}$$

where ω and V are the angular frequency and amplitude of rf, r_0 is the radius of the resonator cross section, $J_k(r)$ denotes the Bessel function of kth order, and $\zeta_{k\ell}$ is the ℓth root of the algebraic equation $J_k(r) = 0$. We see that a third-order coupling can be developed in TM_{010}-mode operation. In fact, when $r / r_0 \ll 1$, the longitudinal component of Eq. (6) can be approximated by

$$A_z \approx \frac{V}{\omega} \left[1 - \left(\frac{\zeta_{01}}{2} \frac{r}{r_0} \right)^2 \right] \sin(\omega t) \delta_p(s), \tag{7}$$

indicating that A_z depends quadratically on the transverse coordinates. Since time t is regarded as the longitudinal canonical coordinate in the present coordinate system where the path length s is the independent variable, the potential (7) yields a third-order synchro-betatron coupling corresponding to $(m, \ n) = (2, \ 1)$ in Eq. (2). It is indeed possible to provide different types of nonlinear coupling by using other excitation modes.

APPLICATION

Whenever the emittance of a particular direction is more important than those of the other directions or emittances ratios rather than their magnitudes themselves play an essential role, the present idea merits attention. Among a wide range of choices, we here briefly study FELs as a possible application of the coupling ring. After a proper emittance-transfer procedure, the electron beam from the ring is injected into a short transport channel to match the Twiss parameters to a subsequent FEL system. In what follows, we employ the simulation code "GENESIS" [8] to figure out the usefulness of a coupling storage ring. The FEL parameters assumed here are summarized in Table 1.

Table 1: FEL design parameters

Radiation wavelength λ [nm]	6
Beam energy [GeV]	1
Beam current (peak) [A]	250
Undulator period λ_w [cm]	2
Undulator parameter K	1.14

Emittance Optimization for FEL

In order to attain high power from FEL, the current of the electron beam should be as high as possible. In addition, the following conditions are generally required for the beam quality [9]:

(a) The transverse normalized emittances must be less than $\lambda/4\pi$ where λ is the wavelength of radiation.

(b) The energy spread must be less than the FEL parameter ρ.

These requirements indicate that a possible performance of FEL is critically limited by the condition of an incident electron beam; even if the 6D volume is maintained, the FEL gain can be reduced considerably unless the ratios of three projected emittances are properly chosen.

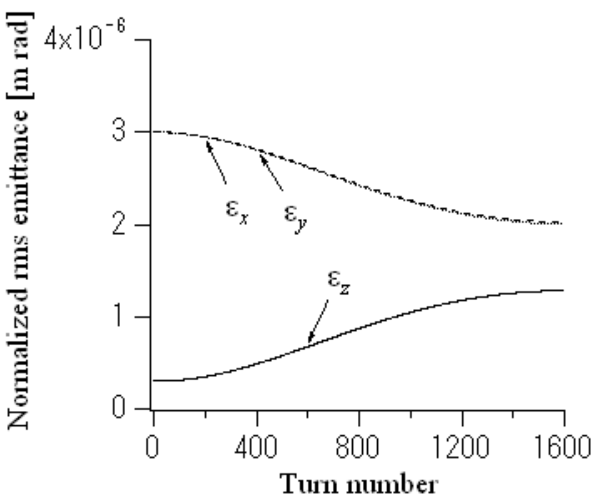

Figure 4: Nonlinear emittance transfer in a coupling storage ring where $(\nu_x, \nu_y, \nu_z) = (3.55,\ 3.55,\ 0.1)$.

The recent progress in accelerator technologies has made it feasible to produce an electron beam with a very low longitudinal emittance [2,3]. The FEL performance is then limited by the transverse beam properties rather than the longitudinal. If the energy spread is much smaller than ρ, the system can tolerate some longitudinal emittance growth with no reduction of the FEL gain. It should then be advantageous to transfer a portion of the phase-space volume from the transverse to the longitudinal direction, so that the condition (a) is more securely fulfilled. Figure 4 shows an emittance exchange process in a coupling ring where a third-order potential ($m=2$, $n=1$) is switched on. The transverse projected emittances have been reduced from 3μm to 2μm after a emittance exchange, while the longitudinal emittance becomes three times greater. The results of GENESIS simulations based on the electron beams before and after this emittance transfer are displayed in Fig. 5 that clearly demonstrates the advantage of the emittance manipulation. Some numerical results concerning linear emittance transfer can also be found in Ref. [4].

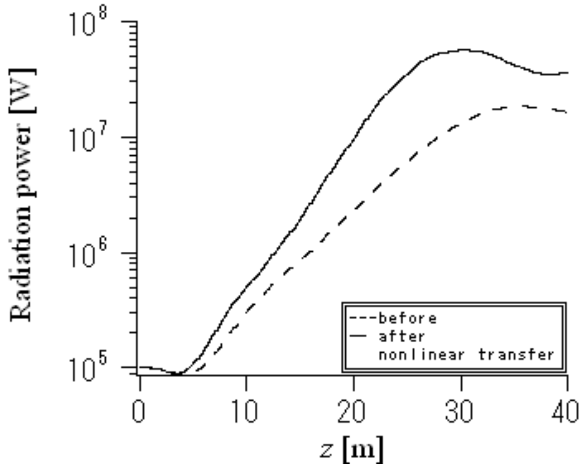

Figure 5: Results of GENESIS simulations. The broken curve was obtained with an original electron beam whose rms emittances are $(\varepsilon_x, \varepsilon_y, \varepsilon_z) = (3\,\mu\text{m}, 3\,\mu\text{m}, 0.3\,\mu\text{m})$. As shown in Fig. 4, this emittance combination is altered, after a nonlinear emittance exchange, to $(\varepsilon_x, \varepsilon_y, \varepsilon_z) =$ $(2.09\,\mu\text{m}, 2.06\,\mu\text{m}, 1.23\,\mu\text{m})$ with which the solid curve was obtained. The power of the seed laser, which is not essential to the present study, is 10^5 W.

Optimum Correlation for FEL

As is well-known, the resonant wavelength λ is determined ideally by $\lambda \approx (1+K^2)\lambda_w/2\gamma^2$ where λ_w and K are the spatial period and normalized strength of the undulator. In reality, however, the beam has a finite energy spread and, furthermore, individual electrons execute betatron oscillations that can modify their average longitudinal velocities. Incorporating the effect of the betatron amplitude in the resonance condition, we obtain

$$\lambda \approx \frac{1}{2}\left(\frac{1+K^2}{\gamma^2} + \frac{2J_x}{\beta_x} + \frac{2J_y}{\beta_y}\right)\lambda_w, \tag{8}$$

where $J_{x(y)}$ denotes the transverse action, and $\beta_{x(y)}$ is the betatron function. This condition implies that, even if the energy of an electron is deviated from the design value by the amount of $\Delta\gamma$, it can still be on resonance with the laser, provided [10]

$$\Delta\gamma/\gamma \approx \kappa_x J_x + \kappa_y J_y, \tag{9}$$

where $\kappa_{x(y)}$ is a constant often referred to as the "conditioning" parameter. Assuming the parameters in

Table. 1, Eq. (9) requires the conditioned phase space as depicted in Fig. 6(a) [10]. Such a special correlation can roughly be established in a coupling ring by exciting nonlinear sources. For instance, let us again take a third-order situation considered in Fig. 4. It is then possible to develop a correlation as shown in Fig. 6(b), by intentionally applying a mismatch to an initial beam. The resultant distribution is not highly conditioned but clearly has a preferable correlation between the energy spread and transverse actions (unlike an electron beam provided by the linear scheme in Ref. [3] where the projected emittances are not partially but only fully exchanged and no correlation remains). Results of GENESIS simulations plotted in Fig. 7 actually confirm that the FEL performance can be considerably improved by such an approximate conditioning procedure.

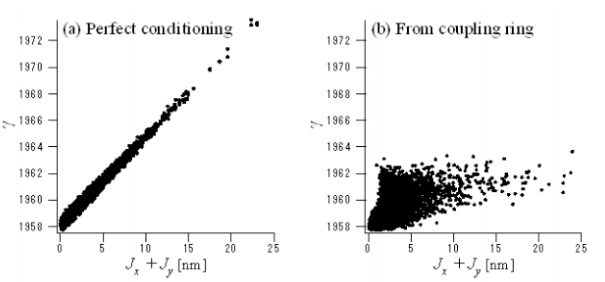

Figure 6: Phase-space correlations corresponding to the perfectly conditioned beam (left) and to an approximately conditioned beam extracted from a third-order coupling ring (right). The same FEL parameters as listed in Table 1 have been taken into account.

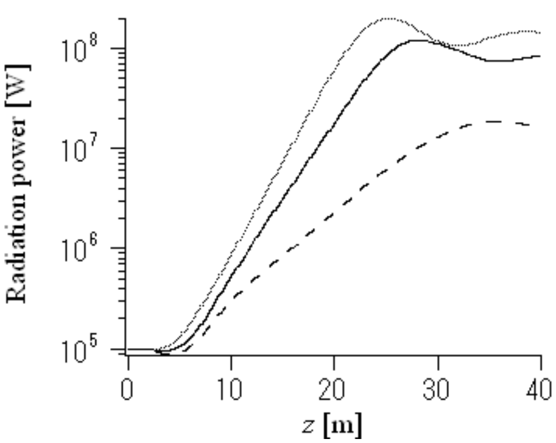

Figure 7: Results of GENESIS simulations. The broken curve was obtained with an original unconditioned electron beam, while the solid curve with the correlated beam shown in Fig. 6(b). The dotted curve is the evolution of the FEL power achievable with the perfectly conditioned beam in Fig. 6(a).

SUMMARY

We have studied a method of manipulating a charged-particle beam in 6D phase space. The present scheme is based on a dedicated storage ring containing linear and nonlinear coupling sources through which the projected emittance of one direction can be transferred to the other directions. The emittance transfer becomes most efficient when the ring operates on coupling resonances. Since the speed of an emittance flow is controllable with lattice parameters such as the coupling constant and tunes, the ratios of projected emittances can be well optimized for diverse purposes. Another important feature of this method is the automatic development of various correlations between degrees of freedom, which may open up a possibility of controlling not only emittance ratios but also even details of the particle distribution in 6D phase space.

As a typical application of the coupling storage ring, we considered FELs where the achievable radiation power is quite sensitive to the transverse and longitudinal projected emittances of an electron beam. Numerical simulations demonstrated that the FEL performance could be improved by a partial emittance transfer. It was also pointed out that the use of a nonlinear coupling enables us to correlate the transverse action with the energy spread of a beam. Although the beam conditioning achieved in the present simulation was not perfect, the resultant FEL gain actually got better compared to an unconditioned case. A further study is, however, needed incorporating detailed lattice structures and more accurate effects of coupling cavities.

ACKNOWLEDGMENT

One of the authors (K.K.) is indebted to Drs. G. Penn, W. M. Fawley, and S. Reiche for valuable comments on FEL designs and GENESIS simulations.

REFERENCES

[1] E.D. Courant, “Conservation of Phase Space in Hamiltonian Systems and Particle Beams”, in Perspectives in Modern Physics, edited by R.E. Marshak (1966).

[2] K.-J. Kim and A. M. Sessler, in *AIP Conf. Proc.* **821**, edited by S. Nagaitsev and R. J. Pasquinelli (2006) 115.

[3] P. Emma, Z. Huang, K.-J. Kim and P. Piot, Phys. Rev. ST Accel. Beams **9** (2006) 100702.

[4] H. Okamoto, K. Kaneta and A. M. Sessler, J. Phys. Soc. Jpn. **76**, No. 7 (2007) 074501; H. Okamoto, unpublished report (February, 1995).

[5] M. Cornacchia and P. Emma, Phys. Rev. ST Accel. Beams **5** (2002) 084001.

[6] H. Okamoto, A. M. Sessler and D. Möhl, Phys. Rev. Lett. **72** (1994) 3977.

[7] H. Okamoto, Phys. Rev. E **50** (1994) 4982.

[8] S. Reiche, “Numerical Studies for a Single Pass High Gain Free-Electron Laser”, Ph.D thesis (Hamburg, 1999).

[9] See, e.g., C. A. Brau, “Free-Electron Lasers”, *Advances in Electronics and Electron Physics* (suppl. 22), published by Academic Press Inc. (1990).

[10] A. M. Sessler, D. H. Whittum and L.-H. Yu, Phys. Rev. Lett. **68** (1992) 309.