FOKKER-PLANCK APPROACH TO THE DESCRIPTION OF TRANSVERSE STOCHASTIC COOLING

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Abstract

A Fokker-Planck equation for transverse stochastic cooling is presented, based on some simplifying assumptions. The calculation of the drift and diffusion coefficients is derived from the signal theory of the pick-up response and the dynamics of the kicker response. The equilibrium transverse action (emittance) distribution turns out to be exponential. Furthermore there is a special solution of the time-dependent equation for the case that the initial action distribution is exponential, as well. The distribution remains exponential, and the average action decreases exponentially towards equilibrium. The cooling rate is equal to the standard textbook rate.

THEORETICAL PRELIMINARIES

Fokker-Planck Equation

Let $2J$ be the usual one-particle emittance. J is an action variable in Hamiltonian theory. ϕ is the betatron angle, conjugate to J. The betatron phase advance between pick-up and kicker is denoted $\mu_k - \mu_p$. The beta functions are β_p and β_k.

The Fokker-Planck equation describes the evolution of the distribution function $\Psi(J_x, J_y, \delta p/p)$. It is a continuity equation with a flux Φ in action space

$$\frac{\partial \Psi}{\partial t} + \mathrm{div}\Phi = 0 \tag{1}$$

The flux is

$$\Phi = F\Psi - \frac{1}{2} D \ \mathrm{grad}\Psi \tag{2}$$

The drift coefficient F describes the average cooling. It generally has three components, e.g.

$$F_x = \lim_{\tau \to 0} \left\langle \frac{\delta J_x}{\tau} \right\rangle \tag{3}$$

The limit $\tau \to 0$ always leads to physical interpretation problems, as a rule of thumb one might say that no essential change of the distribution function should happen during τ. The diffusion tensor D describes the diffusion, its components are

$$D_{mn} = \lim_{\tau \to 0} \left\langle \frac{\delta J_m \delta J_n}{\tau} \right\rangle \tag{4}$$

The Fokker-Planck equation is frequently used for the calculation of the longitudinal momentum distribution (see [1] and references therein). In this work the transverse case is presented. All details of the calculations of the drift and diffusion coefficients are not given, they can be derived along the lines presented in [1].

Simplifying Assumptions

In the following we make the assumption that longitudinal cooling works independently of transverse cooling, and that the longitudinal momentum distribution is given somehow. Then the transverse cooling is decoupled and both cooling pcrocesses can be described by one-dimensional Fokker-Planck equations.

1. All kickers and pick-ups are placed at locations of zero dispersion.
2. All electrodes can be described by a simple linear response model.
3. There is no overlap between different Schottky bands.
4. Chromaticity is neglected.

Because of the decoupling between phase spaces, we derive here a separate Fokker-Planck equation for horizontal cooling, where the horizontal distribution is simply called $\Psi(J)$. We normalize it to 1. We also need the longitunal distribution $\psi(\delta p/p$ which we define to be normalized to the number of particles N.

SIGNALS AND BEAM RESPONSE

Sensitivity and Single Particle Signals

The sensitivity $S(\Omega)$ is used for the description of the quality of delivering accelerating voltages to a beam paricle in a kicker electrode.

It is the ratio between the effective accelerating voltage $U(\Omega)$ and the voltage $V_k(\Omega)$ at the input port of the kicker:

$$U(x, y, \Omega) = S(x, y, \Omega) V_k(\Omega) \tag{5}$$

S depends on the beam velocity and on the position of the particle with respect to the electrode.

In the following it will be assumed that S is linear in x over the full range of betatron amplitudes.

$$S(x, y, \Omega) = x S'(\Omega) \tag{6}$$

The formalism becomes much more complicated if this simplification is abandoned [2].

If such an electrode is used as a pick-up electrode and if it is reciprocal, a particle with revolution frequency ω produces a signal at the betatron sidebands

$$\omega_{m,\pm} = \omega(m \pm Q_x) \tag{7}$$

where ω is the revolution frequency (which depends on the longitudinal momentum via the well-known relationship

$\delta\omega/\omega = \eta\delta p/p$), and Q_x is number of horizontal betatron oscillations per turn. The signal is

$$U_p(\Omega) = \frac{QeZ_l\sqrt{2J_x\beta_x}S_p'(\Omega)}{4}\sum_{m=-\infty}^{+\infty}\exp\left(i\Omega t_0\right) \times \left[e^{i\mu_x}\delta\left(\Omega-\omega_{m,+}\right) - e^{-i\mu_x}\delta\left(\Omega-\omega_{m,-}\right)\right] \quad (8)$$

where Z_l is the line impedance at the output port.

The horizontal deflection after one pass through the kicker follows from the Panofsky-Wenzel theorem

$$\delta p_x = \frac{Qe}{i\Omega}S_k'(\Omega)V_k(\Omega) \quad (9)$$

p is the particle momentum. For a single kick, the emittance changes by

$$\delta J_x = \frac{Qe}{i\Omega p}S'(\Omega)V_k(\Omega)\sqrt{2J_x\beta_x}\sin\mu_x \quad (10)$$

Voltage Power Density

The spectral voltage density $C(\Omega)$ is the Fourier transform of the voltage autocorrelation function $R(\tau) =< V(t)V(t+\tau) >$. The brackets $<>$ denote the expectation value. For coasting beams the autocorrelation R is independent of t, because V is a 'stationary process'.

With the mean emittance

$$< J >= \frac{\int_0^\infty J\Psi\, dJ}{\int_0^\infty \Psi\, dJ} \quad (11)$$

the Schottky density at each pick up at the transverse harmonics is

$$C_p(\Omega_\perp) = \frac{(QeZ_l)^2\,\omega\beta_p}{16\pi\,|m\eta|}\left|S_p'\right|^2 < J > \psi(\delta p/p) \quad (12)$$

It is proportional to the mean emittance and to the momentum density.

Another statistical signal is due to thermal noise. It is often expressed by an effective temperature:

$$C_n = \frac{1}{2}Z_l k_B T_{\text{eff}} \quad (13)$$

Signal Combination and Amplification

The amplification chain (Fig, 1) consists of the signal combination networks at the pick-ups and kickers and an overall voltage gain.

We assume n_p pick-ups, connected such that they are matched to the beam velocity. The voltage gain after power combination is

$$g_p = \sqrt{n_p} \quad (14)$$

The electronic voltage gain is modeled as

$$G = G_0\exp i\Omega T_{pk} \quad (15)$$

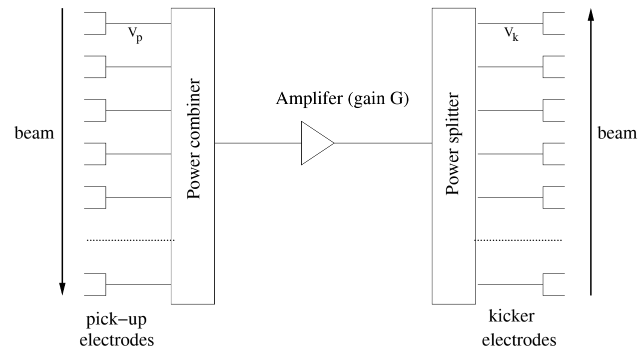

Figure 1: Model of amplification chain.

where G_0 is constant inside the cooling band. T_{pk} is adjusted to the time of flight between pick-up and kicker for design particle. For off-momentum particles

$$\frac{\delta T_{pk}}{T_{pk}} = -\eta_{pk}\frac{\delta p}{p} \quad (16)$$

Here, $\eta_{pk} = \gamma^{-2} - \alpha_{pk}$ must be calculated as a local value from the local momentum compaction between pick-up (at s_p and kicker (at s_k):

$$\alpha_{pk} = \frac{1}{s_k - s_p}\int_{s_p}^{s_k} ds\frac{D(s)}{\rho(s)} \quad (17)$$

where D is the dispersion function and ρ is the radius of the closed orbit.

Then the effective overall accelerating voltage is

$$U_k = xS_k'\, g_k\, G\, g_p\, V_p \quad (18)$$

The effective overall accelerating voltage density becomes

$$C_k = \left|xS_k'\right|^2 |g_k|^2\,|G|^2\left[\left|g_p\right|^2 C_p + C_n\right] \quad (19)$$

TRANSVERSE COOLING

Equations for Transverse Cooling

In the following we use a notation

$$\sum_{m,\pm} = \sum_{m=-\infty}^{+\infty}\sum_{\pm} \quad (20)$$

as we always sum over both sidebands at all harmonics.

The transverse drift then is

$$F_\perp = -\frac{J\,(Qe\omega)^2\,Z_l}{16\pi^2 p}\sqrt{\beta_p\beta_k} \times \sum_{m,\pm}\frac{\pm 1}{\Omega}S_k' g_k\, G\, g_p\, S_p' \times \exp\left[\pm i\left(\mu_k - \mu_p\right) - i\eta_{pk}\,(m \pm Q)\,\omega T_{pk}\delta p/p\right] \quad (21)$$

In order to get a cooling effect, the amplification G must be imaginary, i.e. there must be a 90^0 phase shift in the

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amplification chain. Also, as is well known, the betatron phase shift $\mu_k - \mu_p$ must be an odd integer multiple of $\pi/2$. We define the dimensionless system gain

$$g_\perp = \frac{(Qe)^2\,\omega Z_l}{8\pi p\,\Omega}\sqrt{\beta_p\beta_k}\;S_k'\;g_k\;G\;g_p\;S_p' \tag{22}$$

Note that it is independent of J, but depends on frequency. The definition Eq. (22) differs from the usual textbook definition [1] by a missing factor N as we shall discuss later.

With Eq. (22) the transverse drift becomes

$$\begin{aligned} F_\perp &= \frac{J\omega}{2\pi}\sum_{m,\pm} \pm g_\perp \exp\left[\pm i\left(\mu_k - \mu_p\right)\right. \\ &\times \left. -i\eta_{pk}\,(m \pm Q)\,\omega T_{pk}\delta p/p\right] \end{aligned} \tag{23}$$

We split the transverse diffusion into two terms: one is due to Schottky noise, the second is caused by thermal noise:

$$D_\perp = D_S + D_H \tag{24}$$

With Eq. (12) and Eq. (19) we get for the Schottky term

$$D_S = J < J > \psi(\delta p/p)\frac{\omega}{2\pi}\sum_{m,\pm}\frac{|g_\perp|^2}{|m\eta|} \tag{25}$$

In order get diffusion due to thermal noise, one has to replace the term $\left|g_p\right|^2 C_p$ by C_n.

$$D_n = J\frac{4k_B T_{\text{eff}}}{(Qe)^2\,Z_l\beta_p\left|g_p\right|^2}\sum_{m,\pm}\left|\frac{g_\perp}{S_p'}\right|^2 \tag{26}$$

Altogether the drift and diffusion coefficients can be written in the form

$$F_\perp = -CJ \tag{27}$$

$$D_\perp = J\,(S < J > +H) \tag{28}$$

where the cooling term C, the Schottky diffusion term S and the thermal noise heating term H do not depend on J.

It is worth noting that both coefficients Eq. (27) and Eq. (28) are proportional to J. Therefore, the flux Φ (see Eq. (2)) vanishes at zero emittance. This satisfies the physical boundary condition that there can be no flux toward negative emittances (see Fig. 2).

Transverse Equilibrium Emittance

The condition for equilibrium is

$$-F_\perp\Psi_\infty(J) + \frac{1}{2}D_\perp\frac{\partial\Psi_\infty}{\partial J} = 0 \tag{29}$$

With the coefficients Eq. (27) and Eq. (28) we get

$$\frac{\partial}{\partial J}\left(\ln\Psi_\infty(J)\right) = \frac{\partial\Psi_\infty/\partial J}{\Psi_\infty} = \frac{2F_\perp}{D_\perp} = \frac{-2C}{S < J > +H} \tag{30}$$

This equation is solved by

$$\Psi_\infty(J) = \Psi_0\,\exp\left(-\frac{2C}{S < J >_\infty + H}\,J\right) \tag{31}$$

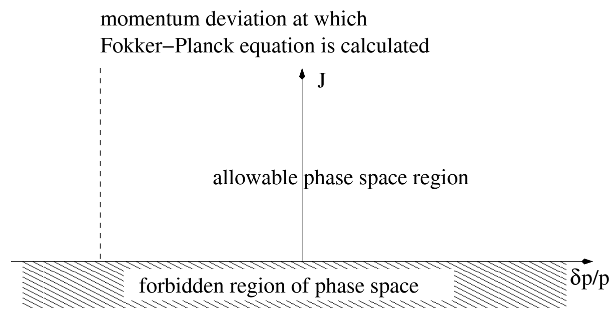

Figure 2: Notation for momentum distribution.

i.e. it is an exponential distribution with average emittance

$$< J >_\infty = \frac{\int_0^\infty J\Psi_\infty\,dJ}{\int_0^\infty \Psi_\infty\,dJ} = \frac{S < J >_\infty + H}{2C} \tag{32}$$

Ψ_0 is a normalizing constant. Solving Eq. (32) for $< J >_\infty$ yields

$$< J >_\infty = \frac{H}{2C - S} \tag{33}$$

The minimum equilibrium emittance is limited by the heating due to thermal noise (if no other heating mechanisms such as intra-beam scattering are taken into acoount). To maintain a normalized solution the condition $2C > S$ is required. Under this condition Eq. (31) is equivalent to

$$\Psi_\infty(J) = \Psi_0\,\exp\left(-\frac{2C - S}{H}\,J\right) \tag{34}$$

with the normalization $\Psi_0 =< J >_\infty$.

Time Dependent Exponential Solution

As the equilibrium transverse distribution is exponential in case of the drift and diffusion coefficients Eq. (27) and Eq. (28), it is reasonable to investigate the following problem: if the initial distribution is exponential, will it remain to be so forever? Hence we try the ansatz

$$\Psi(J,t) = \alpha(t)Ne^{-\alpha(t)J} \tag{35}$$

with the mean emittance

$$< J >= \frac{1}{\alpha} \tag{36}$$

The partial derivatives of the distribution function are

$$\frac{\partial\Psi}{\partial t} = \dot\alpha\left(\frac{1}{\alpha} - J\right)\Psi \tag{37}$$

$$\frac{\partial\Psi}{\partial J} = -\alpha\Psi \tag{38}$$

The Fokker-Planck flux becomes

$$\Phi = F_\perp\Psi - \frac{1}{2}D_\perp\frac{\partial\Psi}{\partial J} = \left(-C + \frac{S + \alpha H}{2}\right)J\Psi \tag{39}$$

Because of

$$\frac{\partial}{\partial J}J\Psi = \alpha\left(\frac{1}{\alpha} - J\right)\Psi \tag{40}$$

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the Fokker-Planck equation $\partial\Psi/\partial t + \partial\Phi/\partial J = 0$ is equivalent to the ordinary differential equation

$$\frac{\dot{\alpha}}{\alpha} + \left(-C + \frac{S + \alpha H}{2}\right) = 0 \tag{41}$$

confirming that Eq. (35) solves the Fokker-Planck equation indeed. In terms of $< J >$ (see Eq. (36)) we get the equivalent equation

$$\frac{d}{dt} < J > + \left(C - \frac{S}{2}\right) < J > - \frac{H}{2} = 0 \tag{42}$$

Because the time derivative in the Fokker-Planck equation is applied only to Ψ, but not to Φ, Eq. (41) remains valid even if the coefficients C, S, and H are *explicitly time-dependent*, as they are if

- the system amplification $g_\perp$ is changed during the cooling process,
- the transverse sensitivities $\partial S/\partial x$ are changed by moving (plunging) the electrodes,
- simultaneous longitudinal cooling is applied, changing the distribution $\psi(\delta p/p)$ (see Eq. (25)).

Equation (42) can be solved analytically with the solution (holding only for constant coefficients)

$$< J > (t) = (J_0 - J_\infty)\, e^{-t/\tau} + J_\infty \tag{43}$$

with the exponential decay time (*not* the instantaneous cooling time)

$$\frac{1}{\tau} = C - \frac{S}{2} \tag{44}$$

and the equilibrium emittance $J_\infty = H/(2C - S)$ which is already known from Eq. (33). Note, however, that Eq. (31) is more general, because the equilibrium distribution always becomes exponential regardless of the initial distribution.

The instantaneous cooling rate is given by the quantity $\dot{\alpha}/\alpha$ of Eq. (41), i.e.

$$\frac{1}{\tau_\perp} = C - \frac{1}{2}S - \frac{H}{2 < J >} \tag{45}$$

This relationship is the base of the following estimates.

COOLING RATE ESTIMATES

General Expression

From Eq. (23), Eq. (25) and Eq. (26) we get

$$\begin{aligned} C &= \frac{\omega}{2\pi}\sum_{m,\pm} \pm g_\perp \exp\left[\pm i\left(\mu_k - \mu_p\right)\right] && (46)\\ &\times \exp\left[-i\eta_{pk}\,(m \pm Q)\,\omega T_{pk}\delta p/p\right] \\ \frac{1}{2}S &= \frac{\omega\psi(\delta p/p)}{4\pi}\sum_{m,\pm}\frac{|g_\perp|^2}{|m\eta|} && (47)\\ \frac{H}{2 < J >} &= \frac{1}{< J >}\frac{2k_B T_{\text{eff}}}{(Qe)^2 Z_l \beta_p \left|g_p\right|^2}\sum_{m,\pm}\left|\frac{g_\perp}{S'_p}\right|^2 && (48)\end{aligned}$$

The first term in this expression is due to the cooling effect in the presence of undesired mixing. The second term is due to Schottky noise heating including the effect of good mixing. The last term is due to thermal noise heating. Equation (46) is still rather general as the amplification term $g_\perp$ is treated as frequency dependent.

Approximations

In order to get a rough estimate of the expected cooling rate, we make the following assumptions:

1. The system gain $g_\perp$ is constant over the system bandwidth W and zero outside. Denote by m_1 and m_2 the harmonic numbers at the lower and upper limits of the cooling band W. The harmonic number at midband is denoted by $m_c = (m_1 + m_2)/2$, the number of harmonics inside the cooling band is $\Delta m = m_2 - m_1$.
2. The betatron phase advance between pick-up and kicker is exactly 90^0.
3. The real part of $g_\perp$ vanishes.
4. The momentum distribution can be approximately written $\psi = N/(\Delta p/p)$. The quantity $\delta p/p$ denotes the momentum deviation where the transverse rate equations are calculated (Fig. 2), whereas $\Delta p/p$ is the total width of the momentum distribution. This quantity is only needed in the Schottky heating term of the rate Eq. (46). It will reappear in the mixing number Eq. (53).

Undesired Mixing

In order to treat the effect of undesired mixing we introduce the angle $\phi_u = \omega T_{pk}\eta_{pk}\delta p/p$ such that the sum in the cooling term of Eq. (46) becomes

$$\begin{aligned} &\sum_{m,\pm} \pm \exp\left[\pm i\frac{\pi}{2} - im\phi_u\right] && (49)\\ =\;& 4i\text{Re}\left(\frac{e^{im_2\phi_u} - e^{im_1\phi_u}}{e^{i\phi_u} - 1}\right) \\ =\;& 4i\frac{\cos\left[\left(m_c - \frac{1}{2}\right)\phi_u\right]\sin\left[\Delta m\frac{\phi_u}{2}\right]}{\sin\left[\frac{\phi_u}{2}\right]} \\ \approx\;& 4i\Delta m\cos\left[m_c\phi_u\right]\end{aligned}$$

We have used the approximation $m \pm Q \approx m$. The last approximation is valid if the undesired mixing is tolerable, i.e. if $\phi_u \ll \pi/(m_2 + m_1)$. To be rigourous we define the undesired mixing number

$$B := \cos\left[m_c\phi_u\right] B_1 \tag{50}$$

where the correction

$$B_1 = \frac{\sin\left[\Delta m\frac{\phi_u}{2}\right]}{\Delta m\sin\left[\frac{\phi_u}{2}\right]} \tag{51}$$

is of the order of unity.

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Optimum Gain and Cooling Rate Estimate

The sum in the Schottky heating expression can be approximately written

$$\sum_{m=m_1}^{m_2} \frac{1}{m} \approx \ln\left(m_2/m_1\right) \tag{52}$$

This gives rise to the mixing number

$$M = \frac{1}{|\eta|\Delta p/p} \frac{\ln\left(m_2/m_1\right)}{m_2 - m_1} \approx \frac{1}{m_c|\eta|\Delta p/p} \tag{53}$$

One gets the last approximation (F. Pedersen, D. McGinnis, priv. comm.) by linearizing the logarithm according to $\ln m_{12} \approx \ln m_c \pm (m_2 - m_1)/2m_c$.

The noise heating term is

$$h = \frac{8k_B T_{\text{eff}}}{Z_l\,(Qe)^2\,\omega\left|g_p\right|^2 \beta_p \left|S_p'\right|^2 < J >} \tag{54}$$

Then we can write instead of Eq. (46)

$$\frac{1}{\tau_\perp} \approx 2W\left[2B\,|g_\perp| - (MN + h)\,|g_\perp|^2\right] \tag{55}$$

Equation (55) looks much like the standard textbook cooling rate, except for the particle number N, which does not appear in front, but only in the desired mixing term M. This is due to a different definition of system gain Eq. (22), which should reasonably not depend on N, because neither the coherent cooling (see Eq. (23)) nor the thermal diffusion (see Eq. (26)) depend physically on the particle number.

The optimum system gain is

$$|g_\perp|_{\text{opt}} = \frac{B}{MN + h} \tag{56}$$

and the optimum cooling rate is

$$\left(\frac{1}{\tau_\perp}\right)_{\text{opt}} = \frac{2WB^2}{MN + h} \tag{57}$$

The quantities C, H and S from Eq. (27) and Eq. (28) are related to B, M and U via the relationships

$$C = 4Wg_\perp B \tag{58}$$

$$\frac{S}{2} = 2W\,|g_\perp|^2\,MN \tag{59}$$

$$\frac{H}{2 < J >} = 2W\,|g_\perp|^2\,h \tag{60}$$

Remarks

- The noise to signal ratio h/N is the noise power at the effective temperature T_{eff} divided by the Schottky power of a beam of N particles with average emittance $< J >$ at one betatron sideband. If the emittance decreases during the cooling process, h can be inhibited from increasing too fast by using plunging electrodes. They have the effect of increasing the electrode responses S_p' and S_k'.
- One should note the $1/\Omega$ dependence in the system gain Eq. (22). If the sensitivity slope S' is constant over the bandwidth, then one has to increase the electronic amplification G by 6 dB per octave in order to get a constant system gain. This is a consequence of the $1/\Omega$-scaling of the transverse kicks strength due to the Panofsky-Wenzel theorem Eq. (9).

CONCLUSIONS

1. We have given explicit expressions for the drift and diffusion coefficients of transverse stochastic cooling.
2. We have expressed the system gain in terms of electronic parameters.
3. We have proved from very general assumptions that the equilibrium emittance distribution is exponential.
4. We have proved that an initial exponential emittance distribution remains exponential during the cooling process.
5. We have presented a general, frequency-dependent cooling rate equation.
6. We have shown that in the case of constant system gain over a given bandwidth, we can reproduce the results of standard stochastic cooling theory.
7. In this case, we have given explicit expressions for the mixing number M and the noise heating term h.

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