CRYSTALLINE BEAM STUDIES WITH ANDY SESSLER*

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Abstract

For more than two decades since 1992, Andy Sessler worked with us as a hobby on the topic of crystallization of charged ion beams and cooling methods. In this paper, we review the studies jointly performed with Andy highlighting major findings and challenges, and discuss current status and possible future topics and directions.

INTRODUCTION

Beam crystallization has been a topic of interests since first evidence of experimental anomaly was observed on an electron-cooled proton beam at the storage ring NAP-M in 1980 [1]. Starting 1985, J. Schiffer and co-workers studied the properties of one-component, non-relativistic charged particles in the external potential of a simple harmonic oscillator using the molecular dynamics method [2]. Since then, strong space-charge dominated phenomena and one-dimensional (1-D) ordering states were reported with both proton and heavier ions at storage rings TSR [3], ASTRID [4], ESR [5], CRYRING [6], and S-LSR [7] (Table 1).

In 1992, A.G. Ruggiero introduced A.M. Sessler and J. Wei (JW) to the discussion of studying crystalline beams in realistic storage ring conditions. The study immediately involved X.-P. Li (XPL) who is a condensed matter physicist by training. We realize that the most straightforward and rigorous approach is to derive the equations of motion in the so-called beam rest frame where the reference particle is at rest. In this frame, the conventional method of condensed matter physics can be readily applied.

It took JW six month to adopt the formalism of general relativity to derive the equation of motion in the beam rest frame using numerical algebra methods [8]. In another month, XPL developed the beam dynamics algorithms as well as other relevant condensed matter analysis algorithms. Together we created the codes SOLID that can be used to rigorously study beams at ultra-low temperature regime [9].

To attain an ordered state, effective beam cooling is needed to overcome beam heating caused by coherent resonance crossing and intra-beam scattering. Furthermore, the cooling force must conform to the dispersive nature of a crystalline ground state in a storage ring for 3-D structures. To reach the state of crystalline beams in numerical simulations, we used artificial cooling methods enforcing periodicity of the particle motion in the beam rest frame. H. Okamoto (HO) led the analysis of actual beam cooling methods including coupled cooling and tapered cooling [10]. Thus, the entire theoretical approach on beam crystallization was developed.

For more than two decades since 1992, Andy worked closely with us on all major topics of beam crystallization from the derivation of equations of motion to numerical simulation and then to realization with practical cooling methods. Figures 1 to 6 show various occasions associated with Andy during the past 20 years. Andy hosted our extended visits to Lawrence Berkeley National Laboratory (LBNL) in formulating the study approaches and identifying major directions of breakthrough (Fig. 1). He led the interaction with major experimental groups at Aarhus, Denmark and Heidelberg, Germany, providing insights in experimental benchmarking (Fig. 2). Starting from 1997, Andy visited Kyoto University, Japan, for extended periods of time stimulating both experimental and theoretical beam cooling and crystallization work in Japan. Later, he stayed at Hiroshima University collaborating with H. Okamoto and his students (Fig. 4).

Andy has been our mentor, role model, colleague, and friend. He was always fresh with new ideas and passionate about physics, life, and friendship. During early years of study at Berkeley, he often came up with a dozen new ideas a day as we explored the fascinating physics of beam crystallization. Even though most did not survive the subsequent trial and error, some most important findings originated from Andy's imagination.

Figure 1: A.M. Sessler in Muir Woods, California, USA, 1993 (photo taken by J. Wei). Andy hosted extended visits to LBNL at multiple stages of beam crystallization studies.

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Table 1: A compilation of experimental parameters and observations at existing storage rings: NAP-M [1], TSR [3], ASTRID [4], ESR [5], CRYRING [6], and S-LSR [7].

	NAP-M	TSR	ASTRID	ESR	CRYRING	S-LSR
E_u [MeV/u]	65.7	1.9	0.00417	360	7.4	7
Circumference [m]	47.25	55.4	40	108.36	51.63	22.557
γ	1.07	1.002	1.00000444	1.384	1.00789	1.00746
γ_T	1.18	2.96	4.34	2.67	2.25	1.23
N_{SP}	4	2	4	6	6	6
$\frac{\nu_x}{N_{SP}} / \frac{\nu_y}{N_{SP}}$	0.338 / 0.315	1.285 / 1.105	0.345 / 0.33	0.383 / 0.383	0.383 / 0.383	0.24 / 0.24
Species	Proton	$^{7}Li^{+}$	$^{24}Mg^{+}$	$^{197}Au^{79+}$	$^{129}Xe^{36+}$	Proton
Cooling Method	EC	LC	LC	EC	EC	EC
ξ [μm]	4.6	4.15	21.8	12.7	11.2	4.82
$T_{Bx,y}$ / T_{Bz} [K]	50 / 1	-- / 3	>0.1 / 0.001	13580.6 / <10	27.2 / 18.1	9.05 / 1.54
$T_{x,y}$ / T_z	13.9 / 0.28	-- / 0.75	>0.132 / 0.00132	1.68 / <0.001	0.014 / 0.009	2.64 / 0.45
N_0 (anomaly)	2×10^{7}	--	5.5×10^{8}	4000	1000—10000	2000
N_0 (1-D to 2-D)	6.0×10^{6}	1.4×10^{7}	1.1×10^{6}	7.9×10^{6}	4.7×10^{6}	2.9×10^{6}
Observations	Schottky anomaly	Indirect transverse cooling	Schottky anomaly	1-D ordering	1-D ordering	1-D ordering

Figure 2: J. Wei, A.G. Ruggiero, A.M. Sessler, J. Hangst, and J. Schiffer (from left to right) at the *31st Workshop of the INFN Eloisatron Project on Crystalline Beams and Related Issues*, Erice, Italy, 1995 (photo taken by R. Ma).

Andy was an avid outdoor person and loved sharing physical activities, such as skiing, hiking, and jogging (Fig. 1). We routinely jog during lunchtime to the "Inspiration Point" at Berkeley, sharing jokes and solving physics challenges. Just as he loved sharing the outdoors, he loved sharing ideas, thoughts, knowledge, and life-time experiences. We went to Andy not only on physics problems but also upon life and career decisions.

Andy believed that physics is means to great friendship. He enjoyed friendship worldwide and across several generations of scientists. Andy frequently hosted students and junior physicists from Asia, Europe, and USA working on diverse subjects of accelerator physics. He loved travels not only to conferences, workshops and meetings in different continents but often on adventure trips to faraway places in the world.

Figure 3: J. Wei, X.-P. Li, K. Takayama and S. Yu (from left to right) at the *Andy Sessler Symposium*, Berkeley, USA, 2003. (From photos.lbl.gov, © Regents of UC through LBL.)

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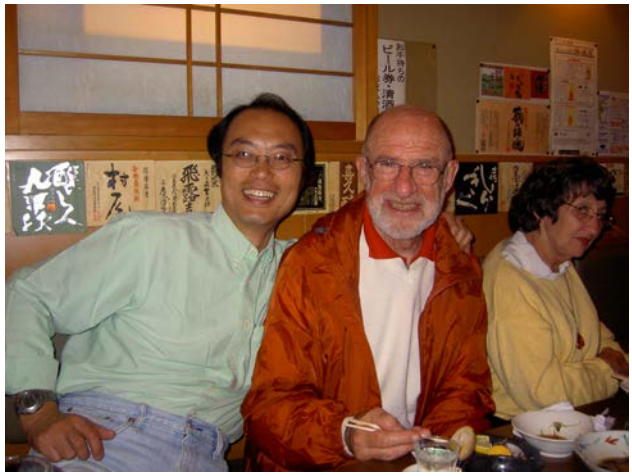

Figure 4: H. Okamoto, A.M. Sessler and S. Adams (from left to right) in Hiroshima, Japan, 2006.

Figure 5: A. Hofmann, A.M. Sessler, E. Wilson, E. Keil, B. Zotter and D. Möhl (from left to right) at CERN, Switzerland, 2009 (photo taken by H. Okamoto).

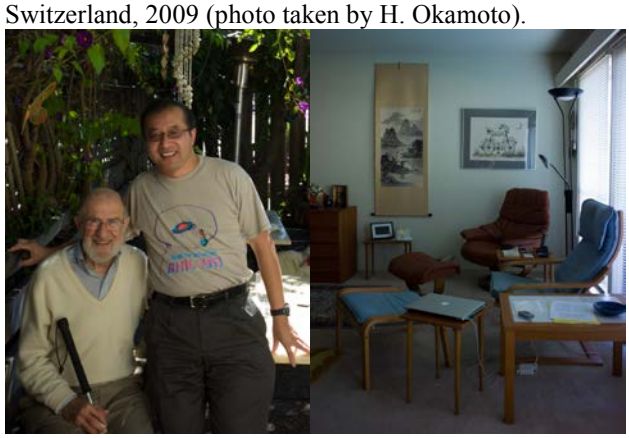

Figure 6: A.M. Sessler and J. Wei discussing work (left) near Andy's apartment (right) at Oakland, USA, October 3, 2013.

Andy has always been optimistic and realistic facing physics problems as well as life situations. Referring to his deteriorating medical situation, Andy wrote on August 16, 2013: "I have had a wonderful life: family, professionally, recreation, travel. Couldn't ask for more. I have taken it as it comes, and in the same spirit I am taking this."[11] When JW visited him in October 2013, six months before his passing (Fig. 6), Andy spent half a day discussing candidates that Michigan State University and the Facility for Rare Isotope Beams (FRIB) Project should approach and recruit. He had strong opinion on the importance of retaining free energy and on the long-range wellbeing of basic, non-programmatic beam sciences.

Upon receiving the Enrico Fermi Award in February 2014, Andy said: "Actually, it has been great fun developing all these things. I had more fun doing that—through the years—than I am enjoying even now on this happy day." "I am pleased that this award recognizes the activity of accelerator and beam scientists. I was fortunate to be in, almost at the beginning, when the power of theoretical physics was first brought to bear on accelerators." [12]

Figure 7: S. Adams, B. Obama and A.M. Sessler at the 2013 Enrico Fermi Award Ceremony in the White House, Washington D.C., USA, on February 3, 2014 [12].

CRYSTALLINE BEAM IN STORAGE RINGS

In this section, we summarize major results obtained during our study on crystalline beams.

For a system of particles with charge Ze and mass Am_0, the characteristic length ξ is given by [9]

$$\xi = \left(\frac{Z_0^2 r_0 \rho^2}{A\beta^2\gamma^2}\right)^{\frac{1}{3}}$$

where r_0 is the proton classical radius, βc and $\gamma A m_0 c^2$ are the velocity and energy of the reference particle, and ρ is the radius of curvature of the main bending region of the storage ring. In a typical accelerator (Table 1) this characteristic length corresponds to a regime where quantum effects are negligible and classical dynamics is adequate [13].

Crystalline Beam Ground State

In the classical dynamics limit, the "ground state" of a crystalline beam corresponds to "zero emittance" state where the motion of the circulating particles is periodic in time with the period of the machine lattice. As shown in

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Fig. 8, particle trajectory in the transverse direction conforms to AG focusing (breathing), and in the longitudinal direction conforms to the change in bending radius (shear). In the presence of a longitudinal electric field, momentum p_z also varies periodically conforming to the energy gain at the cavity.

The goal of beam crystallization is to reach such kind of zero-emittance state where all "betatron" (transverse) and "synchrotron" (longitudinal) motions vanish and where all particles follow the periodic "closed orbit" as the external focusing force and intra-particle Coulomb force balance each other [8,14].

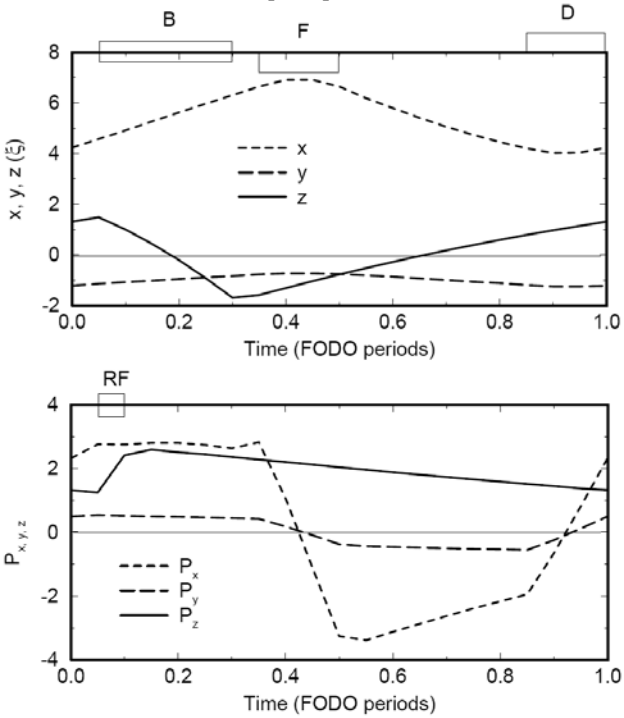

Figure 8: Particle trajectory of a bunched crystalline beam. The machine consists of 10 FODO cells with ν_x=2.8, ν_y = 2.1, and γ =1.4. Lattice components in each cell are displayed on the figure: B is a bending section, F and D are focusing and de-focusing quadrupoles, and RF is the bunching rf cavity.

General Relativity Derivation of Beam Rest Frame Hamiltonian

Although there can be more simplified method, we decided to adopt the general relativity formalism by C. Møller [15] to rigorously derive the equation of motion in the beam rest frame by tensor algebra. For a storage ring consisting of dipole magnetic of field B_0, quadrupole magnet of field gradient $\frac{\partial B_y}{\partial x}$, and accelerating electrical field (E_s), the motion can be represented by the Hamiltonian [8]

$$H_{bend} = \sum_i \frac{1}{2}\left[P_{ix}^2 + P_{iy}^2 + P_{iz}^2\right] - \gamma x_i P_{iz} + \frac{1}{2}[(1-n)x_i^2 + n y_i^2] + V_{ci}$$

for the bending region of radius ρ of the storage ring and

$$H_{non-bend} = \sum_i \frac{1}{2}\left[P_{ix}^2 + P_{iy}^2 + P_{iz}^2\right] + \frac{1}{2}[-n\, x_i^2 + n\, y_i^2] + V_{ci} + U_s$$

for the non-bending region of the ring, where

$$V_{ci} = \sum_{j\neq i} \frac{1}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}}$$

is the Coulomb potential,

$$\mathrm{n} = -\frac{\rho}{B_0}\frac{\partial B_y}{\partial x}, \quad \frac{\partial U_s}{\partial z} = -\frac{Z_0 e \xi E_s}{m_0 c^2}\left(\frac{\rho}{\xi\beta\gamma}\right)^2.$$

Condition for Ground State Existence

The storage ring must be alternating-gradient focusing operating below the transition energy, γ_T:

$$\gamma < \gamma_T.$$

Although some colleagues may argue that this condition is equivalent to avoiding the negative-mass instability [16], we derived it from the very first principle that the beam-frame Hamiltonian is bounded [17]. It arises from the criterion of stable kinematic motion under Coulomb interaction when particles are subject to bending in a storage ring.

Condition to Avoid Phonon Resonances

The bare transverse phase advances per lattice period need to be less than 90°, i.e.

$$\frac{\nu_{x,y}}{N_{SP}} < \frac{1}{4}$$

where ν_x and ν_y are the bare horizontal and vertical tunes, and N_{SP} is the lattice super-periodicity of the storage ring.

Although some colleagues argue that this condition is equivalent to avoiding the envelope instability [18], we again derived it from the very first principle of single-particle dynamics as the criteria that there is no linear resonance between the phonon modes of the crystalline structure and the machine lattice periodicity, and that linear resonance stopbands are not crossed during the entire cooling process as the 3-D beam density is increased.

Ground State Structure

The ground state structure is a 1-D chain when the beam line density is low [19]. The structure becomes 2-D lying in the plane of weaker transverse focusing if the line density λ in the machine is

$$\lambda > 0.62\gamma\xi^{-1}\left[\min\left(\nu_y^2, \nu_x^2 - \gamma^2\right)\right]^{\frac{1}{3}}$$

For even higher density, the particles arrange themselves into 3-D crystals, becoming helices and then helices within helices. Figure 9 shows such a multi-shell structure at a de-focusing location of the lattice. The maximum spatial density in the laboratory frame is approximately $\gamma\nu_y\sqrt{\nu_x^2 - \gamma^2}/(2\xi^3)$. If a sinusoidal electric field is present, the crystalline structure can be bunched azimuthally [18].

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Beam and Lattice Heating

The minimum cooling rate needed for beam crystallization corresponds to the intra-beam scattering heating in an AG-focusing lattice. At any non-zero temperature the beam absorbs energy and heats up under time-dependent external forces caused by variations in lattice focusing and bending. In the high temperature limit, this intra-beam scattering results in a growth rate proportional to $\lambda T^{-5/2}$. The peak heating rate occurs at the temperature of about $T \approx 1$ when the ordering starts to occur, as shown in Fig. 10. Typically, strong spatial correlation appears in all directions when the temperature is below $T \approx 0.05$. Lattice shear and AG focusing have similar effects on beam heating. Heating behavior is similar for both bunched and coasting beams.

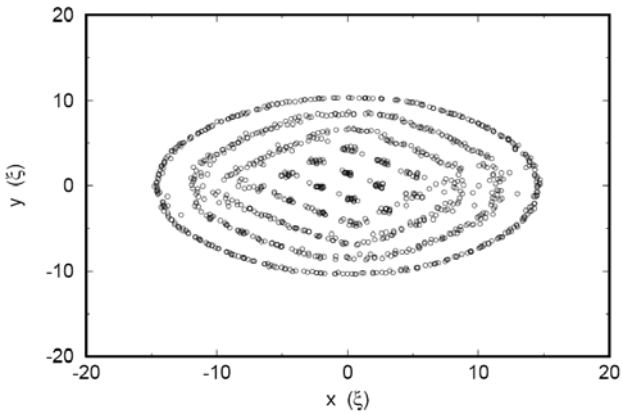

Figure 9: A multi-shell structure with particle positions projected into the x-y plane ($\lambda = 25\gamma\xi^{-1}$).

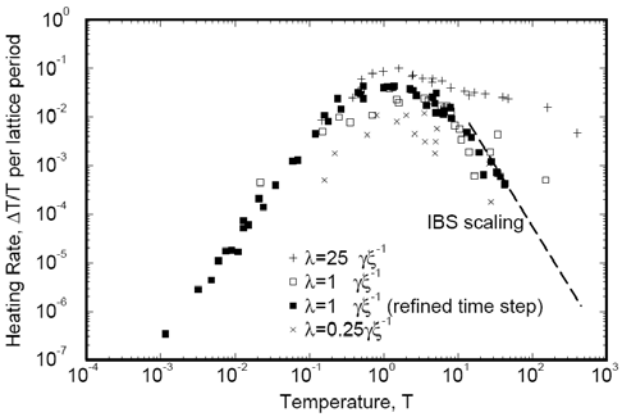

Figure 10: Typical heating rates as functions of temperature obtained by MD simulation at various line density, λ.

The normalized temperature T is related to be conventionally defined beam-frame temperature T_B and laboratory-frame parameters by

$$T_{x,y,z} = \frac{k_B \rho^2}{A m_0 c^2 \beta^2 \gamma^2 \xi^2} T_{Bx,y,z}$$

where

$$[T_{Bx}, T_{By}, T_{Bz}] = \frac{\beta^2\gamma^2 m_0 c^2}{2k_B}\left[\frac{\epsilon_x}{\langle\beta_x\rangle}, \frac{\epsilon_y}{\langle\beta_y\rangle}, \frac{1}{\gamma^2}\left(\frac{\delta p}{p}\right)^2\right].$$

Here, $\epsilon_{x,y}$ are the unnormalized emittances, $\delta p/p$ is the rms momentum spread, $\langle\beta_{x,y}\rangle$ are the average betatron amplitude function of the storage ring, and k_B is the Boltzmann constant.

CONDENSED MATTER METHODS FOR THR BEAM REST FRAME

In this section, we outline the framework to numerically solve the Hamiltonian for crystalline beams in order to study their condensed matter properties. We will conclude with speculations and open questions.

Method

The derivation of the Hamiltonian [8] makes it convenient to study the condensed matter properties of crystalline beams in storage rings. It is straight forward to write a molecular dynamics program to simulate the motion of particles, either to cool down the beam using efficient but unrealistic cooling methods to find ground states, or to employ realistic cooling methods to simulate the dynamics of a crystalline beam.

For condensed matter properties, it is desirable to separate the hopefully smaller time-dependent part in the Hamiltonian from the time-independent part, and then treat the time-dependent part as a perturbation. The time-independent part of the Hamiltonian is obtained by the Smooth Approximation [8].

Recall that with the smooth approximation, the Hamiltonian is

$$H = \sum_i \frac{1}{2}\left[P_{ix}^2 + P_{iy}^2 + (P_{iz} - \gamma X_i)^2\right] + \frac{1}{2}\left[(\nu_x^2 - \gamma^2)X_i^2 + \nu_y^2 Y_i^2\right] + V_{ci}$$

The only part that needs special attention in molecular dynamics or other numerical studies is the long-range nature of Coulomb interaction between the particles, when the beam is uniformly spread throughout the storage ring, that is, not bunched. Periodic boundary conditions in the longitudinal direction can be applied in this case.

Using the notation that L is the size of the periodic cell, $\varrho = \sqrt{X^2 + Y^2}$ is radius in the transverse direction, N is the total number of particles in a cell, λ is the linear density, then the Coulomb energy per particle is

$$\frac{V_c}{N} = \frac{1}{NL}\sum_{j<i} \Phi(\varrho_{ij}, Z_{ij})$$

where Z_{ij} is understood to be modulo L. The detailed algebra to efficiently calculate $\Phi(\varrho, z)$ had been worked out by Avilov [20] and Hasse [21]. Avilov [20] gives

$$\Phi(\varrho, Z) = \frac{4}{L}\sum_{m=1}^{\infty} K_0\left(\frac{2\pi m\varrho}{L}\right)\cos\left(\frac{2\pi m Z}{L}\right) - \frac{2}{L}\left[\ln\left(\frac{\varrho}{2L}\right) + \gamma\right]$$

Where $K_0(r)$ is the zeroth order McDonald function, and γ is the Euler constant. This expansion converges rapidly as long as ρ is not near 0.

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Based on Avilov, Hasse [21] gives another elegant formula:

$$\Phi(\varrho,Z)=\frac{1}{(\varrho^2+Z^2)^{1/2}}-\sum_{\mu=0}^{\infty}\frac{\left(-\frac{\varrho^2}{4}\right)^{\mu}}{(\mu!)^2}\frac{d^{(2\mu)}}{dz^{(2\mu)}}[\psi(Z)+\psi(-Z)+2\gamma]$$

Where

$$\psi(Z)+\psi(-Z)+2\gamma=-2\sum_{n=0}^{\infty}[\zeta(2n+1)-1]Z^{2n}$$

And $\zeta(n)$ is the Rieman zeta function of integer argument. This series converges fast when ϱ is small.
In practice, we can use Hasse's formula for small ϱ and Avilov's formula for larger ϱ.

Ground States

With Smooth Approximation and employing a downhill-type algorithm like conjugate gradient, we can calculate the time-independent ground state structures and energies. A typical ground state energy versus linear density curve is shown in Fig. 11.

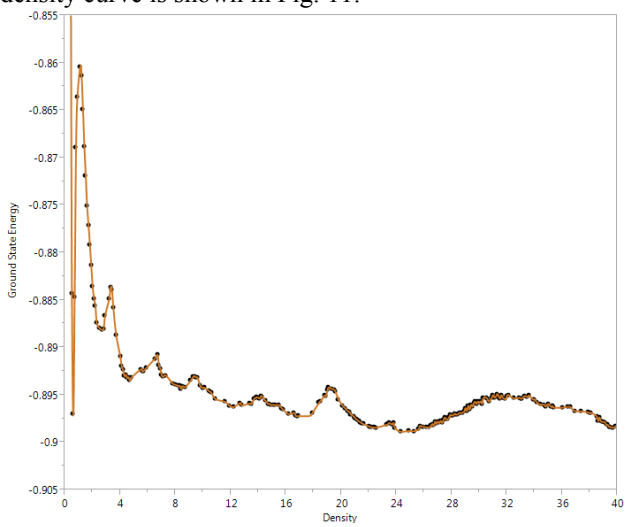

Figure 11: Typical ground state energy versus linear density of crystalline beams in storage rings.

Figure 12 shows some example ground state structures when linear density increases.

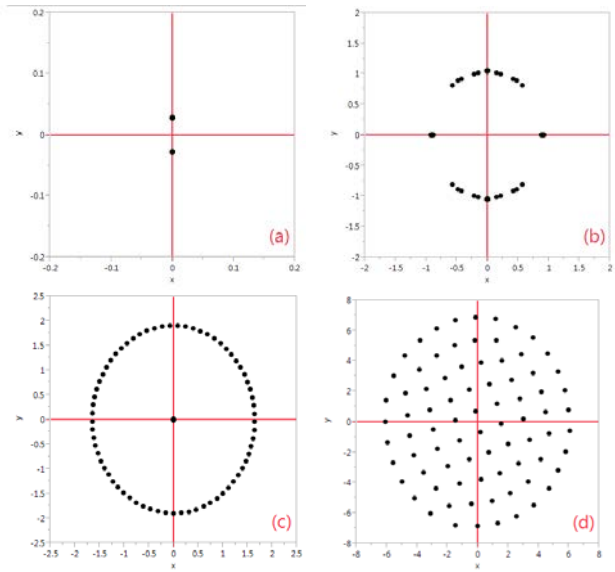

Figure 12: Examples of ground state structures. (a) Zig zag chain; (b) One shell; (c) A second shell starts to emerge in the center; (d) Multiple shells which resembles a lattice near the center.

Phonon Spectrum

Once the time-independent ground state structure is found, we need to find the excited states in order to study the effects of the time-dependent part of the Hamiltonian, as well as finite temperature properties. The low-lying excited states can be approximated by small vibrations of the particles, or phonons. Another important reason to study the phonon spectrum is that, as established earlier, a stabilization criterion is related to the highest phonon frequency [22].

The Coulomb interaction term is the only non-linear term that needs a small vibration expansion. Using capital letters (X,Y,Z) to denote the coordinates of equilibrium positions and $(\tilde{x},\tilde{y},\tilde{z})$ the magnitude of the small deviations from the equilibrium position, applying periodic boundary conditions, the linearized equations of motion are [23]

$$\begin{aligned}\omega^2\tilde{x}_i &= -i\gamma\omega\tilde{z}_i+(\nu_x^2-\gamma^2)\tilde{x}_i\\ &+\sum_{n=-\infty}^{\infty}\sum_{j=1}^{N}\{\left[\frac{1}{R_{nij}^3}-\frac{3(X_i-X_j)^2}{R_{nij}^5}\right]\left[e^{ik(Z_i-Z_j-nL)}\tilde{x}_j-\tilde{x}_i\right]\\ &-\frac{3(X_i-X_j)(Y_i-Y_j)}{R_{nij}^5}[e^{ik(Z_i-Z_j-nL)}\tilde{y}_j-\tilde{y}_i]\\ &-\frac{3(X_i-X_j)(Z_i-Z_j-nL)}{R_{nij}^5}[e^{ik(Z_i-Z_j-nL)}\tilde{z}_j-\tilde{z}_i]\}\end{aligned}$$

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$$\omega^2\tilde{y}_i = \nu_y^2\tilde{y}_i + \sum_{n=-\infty}^{\infty}\sum_{j=1}^{N}\{-\frac{3(X_i-X_j)(Y_i-Y_j)}{R_{nij}^5}\left[e^{ik(Z_i-Z_j-nL)}\tilde{x}_j-\tilde{x}_i\right] + \left[\frac{1}{R_{nij}^3}-\frac{3(Y_i-Y_j)^2}{R_{nij}^5}\right][e^{ik(Z_i-Z_j-nL)}\tilde{y}_j-\tilde{y}_i] - \frac{3(Y_i-Y_j)(Z_i-Z_j-nL)}{R_{nij}^5}[e^{ik(Z_i-Z_j-nL)}\tilde{z}_j-\tilde{z}_i]\}$$

$$\omega^2\tilde{z}_i = i\gamma\omega\tilde{x}_i + \sum_{n=-\infty}^{\infty}\sum_{j=1}^{N}\{-\frac{3(X_i-X_j)(Z_i-Z_j-nL)}{R_{nij}^5}\left[e^{ik(Z_i-Z_j-nL)}\tilde{x}_j-\tilde{x}_i\right] - \frac{3(Y_i-Y_j)(Z_i-Z_j-nL)}{R_{nij}^5}\left[e^{ik(Z_i-Z_j-nL)}\tilde{y}_j-\tilde{y}_i\right] + \left[\frac{1}{R_{nij}^3}-\frac{3(Z_i-Z_j-nL)^2}{R_{nij}^5}\right][e^{ik(Z_i-Z_j-nL)}\tilde{z}_j-\tilde{z}_i]\}$$

Where $R_{nij}=\sqrt{(X_i-X_j)^2+(Y_i-Y_j)^2+(Z_i-Z_j-nL)^2}$ and $i=1,2,\ldots,N$. It is understood that the $R_{nij}=0$ term is excluded from the double sum.

The double sum again has infinite number of terms, which seems to call for some algorithm, like the Ewald Summation, to perform it efficiently. However, considering that all the terms are in the order of $\frac{1}{R_{nij}^3}$ which is short ranged, and that modern computers are fast, it is probably easier to perform the sum by brute force to whatever accuracy required.

Due to the long-range nature of the Coulomb interaction and the fact that this system is quasi one dimensional, the phonon spectrum has a singularity at $k=0$. The singularity is very week, only logarithmic in nature, in the order of $k(\log(k))^{1/2}$, and does not seem to cause anything interesting.

There is another somewhat significant difference between this phonon problem and traditional phonon problems – the frequency ω appears in the off-diagonal of the matrix. This is caused by the cross term between x and p_z in the Hamiltonian. Fortunately when ω is real, the matrix is still Hermitian as expected. Because of this difference, a little extra work is needed to solve this matrix numerically compared to typical eigenvalue problems. We found that it is reasonably easy to solve this whole phonon problem by finding one ω at a time, iteratively until self-consistency.

An example of phonon dispersion curves is shown in Fig. 13, and typical phonon densities of states are shown in Fig. 14.

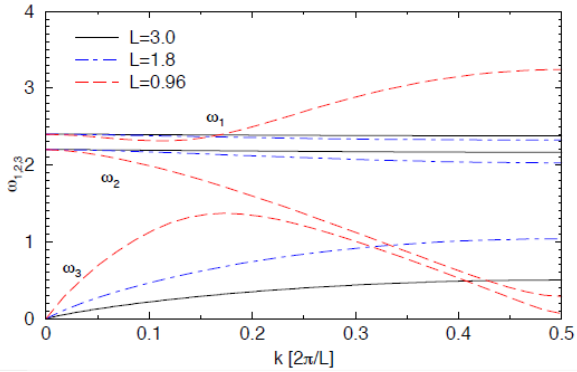

Figure 13: Example phonon dispersion curves.

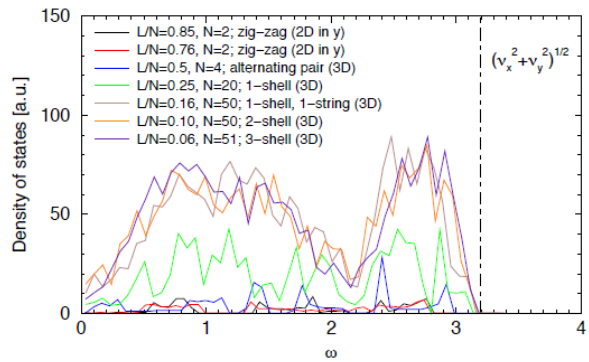

Figure 14: Example phonon densities of states.

Other Considerations

Is there a phase transition? We have established earlier that in order for a crystalline beam to possibly exist, the storage ring has to satisfy $\gamma < \nu_x$. What happens near $\gamma = \nu_x$?

The partition function $Z = Tr(e^{-\beta H})$ for the Hamiltonian under smooth approximation can be written as

$$Z = \left(\frac{2\pi}{\beta}\right)^{\frac{3}{2}N}\left(\frac{2}{\beta}\right)^{\frac{3}{2}N}\left(\frac{1}{(\nu_x^2-\gamma^2)\nu_y^2}\right)^{\frac{N}{2}}\mathbb{Z}(\beta,\nu_x^2-\gamma^2,\nu_y^2)$$

Where $\mathbb{Z}(\beta,\nu_x^2-\gamma^2,\nu_y^2)$ is a massive integration too hard to calculate.

The mean square displacement in the x direction is

$$\langle X^2\rangle = -\frac{2}{\beta N}\frac{\partial \ln(Z)}{\partial \nu_x^2} = \frac{1}{\beta(\nu_x+\gamma)(\nu_x-\gamma)} - \frac{2}{\beta N}\frac{\partial \ln\left(\mathbb{Z}(\beta,\nu_x^2-\gamma^2,\nu_y^2)\right)}{\partial \nu_x^2}$$

Here is a leap of faith – assuming that the last term is analytic near $\gamma = \nu_x$ at $T=0$. The first term shows that there is a second order phase transition at $\gamma = \nu_x$ with a critical exponent 1. However, this is pure speculation. What the last term does is anyone's guess, and temperature effects are also unknown. We do know that $\mathbb{Z}$ arises from the Coulomb interaction. If Coulomb interaction can be neglected, like at very high temperature

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and low density, then $\mathbb{Z}$ is a constant, and the last term in $\langle X^2 \rangle$ is 0, and the phase transition we see here is exact.

Reconcile with von Hove's Theorem. von Hove's theorem [24] is understood to mean the general non-existence of phase transitions in 1D systems with short-range interaction. A beam in a storage ring is a quasi 1D system (it only goes to infinity in the longitudinal direction, yet not zero in the other two dimensions) with long-range Coulomb interaction, thus does not fall to the category of von Hove's theorem. However, it is helpful to understand the possible phase transition if we exam the argument underlining von Hove's theorem.

One of the arguments is given by Landau and Lifshitz [25]. The energy cost of a domain wall is finite, and the entropy caused by the number and locations of the domain walls is $\propto \ln(L)$ where L is the size of the system, therefore the total free energy decreases when the number of domain walls increases at any finite temperature, until no macroscopic domains exist.

In our quasi 1D system, the effect of the long-range Coulomb interaction on the energy cost of a domain wall is beside the point. We observe that in the possible phase transition, the "domain wall" is parallel to the longitudinal direction, thus the energy cost of such a "domain wall" is $\propto L$. So the formation of such a "domain wall" is unfavorable, at least at low temperatures.

"Magic Densities". In private communications with Andy, we raised the question, are there magic densities where crystalline beams are more likely to be observed?

The question arises from the observation that the ground state energy versus density curve has peaks and valleys. Near the peaks, it is less favorable for the density to stay constant throughout the storage ring, as depicted in a (unrealistic) example in Fig. 15. It is therefore more likely to find crystalline beams at densities near the minima of the curve.

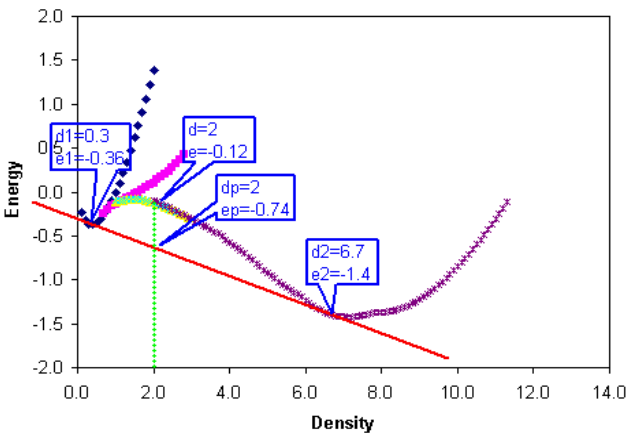

Figure 15: Possible "magic densities" if the ground state energy versus density curve has peaks and valleys. In this illustrative picture, two minima are assumed to be at densities d1 and d2. For a density d between d1 and d2, it is favorable to have a mixture of densities d1 and d2 rather than a uniform density d.

TOWARDS ULTRA LOW TEMPERATURE

Needless to say, it is important to ask whether any realistic cooling method is available to reach a crystalline ground state in practice. The Coulomb coupling constant Γ, defined as the ratio of the average Coulomb energy to the average thermal energy of a beam, is well above 100 in an ideal crystalline state (while $\Gamma \sim 1$ in any regular beams). This typically corresponds to a temperature of a mK range, very close to the absolute zero! In the early 1990's, we only had few cooling methods technically well-established and applicable to hadron beams, i.e., "electron cooling" [26] and "stochastic cooling" [27]. These cooling techniques were, however, insufficient for our ultimate goal because of the achievable temperature much higher than mK. Andy thus paid attention to "Doppler laser cooling" [28,29]. Laser cooling was relatively new to the accelerator community and, as a matter of fact, only two European teams had just begun to apply this sophisticated technique to fast circulating ions in storage rings. The TSR group of Max Planck Institute [3] and the ASTRID group of Aarhus University [4] soon succeeded in demonstrating the promising potential of laser cooling experimentally. Theoretically, the Doppler cooling limit reaches the mK range or even lower, so we naturally concluded that this technique should be the only means for us to approach a crystalline state.

Resonance Cooling

We, however, immediately encountered a serious problem. We learned that laser cooling is effective only in the direction of beam propagation when ions are running at high speed along a particular orbit; no direct cooling force is obtainable in the directions perpendicular to the beam orbit because it is very difficult to ensure a sufficient spatial overlap between a tiny laser spot and fast traveling ions. Dieter Möhl joined us in 1993 to solve this difficulty and, a few months later, we wrote up the idea of using synchro-betatron coupling resonance to indirectly enhance the transverse cooling efficiency [10]. This simple idea was eventually employed at the storage ring S-LSR where a Japanese team successfully confirmed the expected transverse indirect laser-cooling effect under a coupling resonance condition [30,31]

Tapered Cooling

In an early stage of systematic MD simulations, we noticed that too strong a longitudinal linear frictional force could worsen the stability of a large (shell) crystalline configuration [14]. This effect is peculiar to a storage ring that has the momentum dispersion induced by bending magnets. In March 1997, we got together in Kyoto for a couple of weeks, made lots of discussion, and later published a couple of papers summarizing our thoughts at that moment [14,32]. It is well known that multi-shell Coulomb crystals can readily be produced in a compact ion trap with a Doppler cooling system. By contrast, we found it almost hopeless to stabilize such a

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large 3D crystalline beam without the so-called tapered cooling force. Our MD results so far have strongly supported this conclusion [33]. We currently believe that the existence of momentum dispersion in a regular storage ring is one of the most serious obstacles to beam crystallization. In order to overcome the dispersive heating effect, an optimized tapered force must be developed. The effect of cooling is described by a reduction in P_z in the cooling region by

$$\Delta P_z = -f_z(P_z - C_{xz}x)$$

where the coefficient describing the strength of cooling is f_z, and the coefficient describing the extent of tapering C_{xz} is proportional to the lattice dispersion. Alternatively, attempts have been made to design storage rings with dispersion eliminated all around the ring [34]. For this purpose, a unique bending element has been proposed and theoretically studied [35]. Shear-free ring lattices consisting of both magnets and electrodes were designed at S-LSR so that 3-D crystalline structures may be formed without using tapered cooling forces.

Machine Lattice Periodicity Consideration

The condition to avoid phonon resonance discussed in Section 2 is equivalent to avoiding the transverse collective instability driven by the periodic nature of the external alternating-gradient potential for beam focusing. No such instability occurs in a uniform focusing channel. We have repeatedly emphasized the importance of this effect from the very beginning of our fruitful collaboration because it imposes severe restrictions on the lattice design of a dedicated cooler ring. The betatron phase advance per lattice period has to be sufficiently small (most preferably, less than 90 degrees) to avoid crossing a strong linear resonance stop band during a cooling process toward beam crystallization [14,18,33,36,37]. This requirement is very difficult to meet in practice. As any machine involves finite mechanical errors in reality, the original lattice symmetry is always weakly broken. Even if all lattice elements were constructed perfectly and placed precisely at ideal positions, additional coupling sources introduced to enhance transverse laser-cooling efficiency inevitably disturbs the lattice in an asymmetric way. We have confirmed that such weak symmetry breaking can destroy or, at least, affect a large crystalline structure [33,35]. Even the dissipative interactions with laser photons every turn can deteriorate the crystal stability unless the cooling lasers are applied to the beam in all straight sections to strictly hold the symmetry of external forces. These facts suggest that the stabilization of a multi-shell crystalline beam in a storage ring is still quite challenging, although it seems possible to form and maintain the ordered configuration in a carefully designed cooler ring by keeping the cooling lasers on [38]. On the other hand, many past MD results have also shown that the production of the 1D string or 2D zigzag configuration is probably possible even in an existing storage ring. Most recent MD simulations have actually indicated the feasibility of stable string formation of ions in S-LSR with optimized lattice and laser parameters, despite that the ring is equipped with only a single, relatively low-power laser [39].

DISCUSSIONS AND SUMMARY

The crystalline beam corresponds to the ultimate state of zero temperature and zero emittance of charged particle beams. Andy was fascinated by the rich and challenging physics and worked with us as his hobby for more than twenty years. Our contributions range from the fundamental analytical formulation leading to guiding conditions of crystalline beam formation, to numerical methods and confirmation, and then to advising experimentalists in practical realization.

Among practical purposes of beam crystallization, ordered multidimensional beams were proposed for ion-ion colliders for increased luminosity [40]. Machine lattices of high or imaginary transition energy were proposed so that high-energy or colliding crystals may be realized in storage rings of moderate circumference. High-density 1-D strings were proposed for high-luminosity ion-electron collisions with rare radioactive ions [41, 42].

Despite for the efforts made during the past three decades, only 1-D ordering was realized experimentally in storage rings using electron cooling or laser cooling. Higher density, 3-D crystalline structures were only realized in ion traps; efforts to form 3-D crystalline beams in storage rings have not been successful.

Major challenges in beam crystallization are to design and construct storage rings with high lattice periodicity and low transverse phase advance to avoid linear resonances, and to implement effective beam cooling that conforms to the dispersive nature of the beam.

Ion traps have been used to experimentally simulate features of an AG-focusing storage ring [43]. However, it is difficult to study with such a set up the shear effects of the bending magnets in storage rings [44,45]. Combination of a storage ring and an ion trap may simulate the environment of colliding crystals.

Some fundamental questions remain to be answered. Crystalline beam corresponds to a new state of matter of one-component plasma where particles are confined by a periodic, time-dependent external potential with finite transverse boundary. Basic condensed-matter physics of such a system including phase-transition properties remains to be studied.

Since a 3-D crystalline beam has not been observed, the most fundamental question is, does a crystalline beam at a higher density (other than a 1-D string) really exist? If not, why? If yes, under what conditions? There are many variables in the equation, a thorough theoretical understanding and then a comprehensive numerical study of the phase diagram would be most helpful. This is apparently not an easy task. We envision that the first realistic step is to fully understand the already observed phase transition at very low density. It seems to be first order in nature, and progress has been made for its understanding. But why does the transition happen at that

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density? Is it related to the phase transition near $\gamma = \nu_x$ that we speculated? Hopefully an understanding of this transition will have some predicting power and shed lights on higher density conditions.

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