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STOCHASTIC COOLING AS WIENER PROCESS

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Abstract

Traditional theoretical description of stochastic cooling process involves either ordinary differential equations for desired rms quantities or corresponding Fokker-Planck equations. Both approaches use different methods of derivation and seem independent, making transition from one to another quite an issue, incidentally entangling somewhat the basic physics underneath. On the other hand, treatment of the stochastic cooling as Wiener process and starting from the single-particle dynamics written in the form of Langevin equation seems to bring more clarity and integrity. Present work is an attempt to apply Wiener process formalism to the stochastic cooling in order to have a simple and consistent way of deriving its well-known equations.

INTRODUCTION

There are two traditional approaches for theoretical description of the stochastic cooling process – studying parameter evolution of either a single particle or a particle distribution function [1].

The single particle approach involves ordinary differential equations for the rms-particle (i.e. having rms value of a given parameter). The equations are derived by calculation of the first two moments of the kick for a random particle. The cooling process is then described with a coherent effect, which is a particle's own signal, and incoherent effect, which includes all noises for the particle.

The other approach involves Fokker-Planck equations for the particle distribution functions. The derivation is either straightforward and based on the continuity equation analogues to the usual drift-diffusion equation derivation [2] or thorough and based on basic kinetic equations involving all other particle interactions with following simplifications [3]. In this case the cooling is described with quite similar coherent and incoherent terms, which are introduced as drift and diffusion coefficients of the Fokker-Planck equation.

The approach, involving treatment of particle distribution functions over given parameters, is more appropriate for the stochastic cooling simulation, unless we are interested in the initial cooling time or fast draft calculations. Nevertheless, single particle approach is the main tool for the betatron cooling simulation, since the diffusion term is defined by longitudinal dynamics and in this case could be considered constant (or defined by a function).

The connection of coherent and incoherent effects between different approaches was mentioned casually in [4], but it was never explicitly used. Eventually each approach requires its own derivation of coherent and incoherent terms. But both single particle and particle distribution function descriptions use the same underlying model of the cooling process, which involves particle beam, accelerator and stochastic cooling system. This process appears to be a continuous Wiener process (or Brownian process), and corresponding formalism could be immediately applied to the stochastic cooling, giving a straightforward and clear way of deriving the equations and its' coefficients.

LANGEVIN EQUATION

Consider an ensemble of non-interacting particles orbiting in an accelerator and undergoing a stochastic cooling. We are interested in the evolution of some parameter x (momentum spread, emittance, rms betatron amplitudes, etc.) of an arbitrary particle under influence of stochastic cooling system. On each revolution every particle receives a correction kick, or parameter change, from the cooling system, that is the sum of the self-signal of that particle (coherent signal x_c) and some random noise signal due to signals from other particles and noises in the electronics (incoherent signal $\widetilde{x_{ic}}$):

$$\Delta x_{kick} = x_c + \widetilde{x_{ic}}.$$

Since particle parameter depends solely on its present state and kick's interval (revolution period) in most cases could be considered much smaller than cooling time ($T_0 \ll \tau_{Cool}$), the process of stochastic cooling is a continuous Wiener process and all related formalism could be directly applied.

The starting point is then a derivation of a corresponding Langevin equation. The drift F and diffusion D coefficients could be defined in a usual way as:

$$F(x,t) = f_0 \Delta x_c(x,t),$$
$$D(x,t) = 1/2 f_0 \Delta x_{ic}^2(x,t),$$

where $\Delta x_{ic}^2 = \langle \Delta x_{kick}^2 \rangle$, f_0 – revolution frequency.

Then for the given model of stochastic cooling process with non-constant diffusion the corresponding Langevin equation will have the following form [5]:

$$\frac{dx}{dt} = F + \frac{1}{2}\frac{\partial D}{\partial x} + \sqrt{D}\xi(t), \tag{1}$$

where $\xi(t)$ represents Gaussian white noise with the following statistics:

$$\langle \xi \rangle = 0,$$
$$\langle \xi(t)\xi(t') \rangle = 2\delta(t-t').$$

The summand with diffusion derivative in the Equation (1) is needed to compensate the effect of non-constant diffusion, a so-called *noise-induced drift*, which will be introduced later. The logic behind is the same as in explanation of Fick's law of diffusion, some additional details could be found in [5]. The Equation (1) could be used for tracking simulations in a software like Betacool in order to include

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different effects like IBS or electron cooling altogether in the similar fashion.

TIME-AVERAGED DIFFUSION

The standard procedure for deriving the equations within single-particle approach is to calculate the second moment of the kick, such treatment could be found for example in [4]. Here is given simpler, but not that mathematically strict derivation.

We anticipate that the incoherent effect for a given particle has a following statistics:

$$\langle x_{ic} \rangle = 0$$
$$\langle \widetilde{x_{ic}}^2 \rangle = \Delta x_{ic}^2$$

So, on the long-term average we expect that:

$$\frac{d}{dt}x^2 = \overline{\left(\frac{1}{2}\frac{\partial D}{\partial x} + \sqrt{D}\xi(t)\right)^2} = f_0 \Delta x_{ic}^2 = 2D.$$

By processing the derivative, formula for the incoherent dynamics is derived:

$$\frac{dx}{dt} = \frac{D}{x}.$$

Summing up coherent and incoherent effects for the single particle the following equation is derived:

$$\frac{dx}{dt} = F + \frac{D}{x}. \tag{2}$$

Such derivation is valid, because the white noise is considered. Equation (2) could be rewritten for the rms-particle (at a given time) in the more traditional form, involving cooling time $\tau(t)$:

$$\frac{1}{\tau} = -\frac{1}{x_{rms}}\frac{dx_{rms}}{dt} = -\frac{F}{x_{rms}} - \frac{D}{x_{rms}^2}.$$

Eventually, the single particle approach is completely described by the Equation (2). As a simple example, it can be shown that Equation (2) for oversimplified problem coincide with the well-known time-domain formula [4]. Consider:

- Flat distribution of N particles
- $\Delta x_c = -\lambda x$, coherent correction is proportional to the particle's parameter value
- $\Delta x_{ic}^2 = \lambda^2 \cdot x^2 N_s + \lambda^2 \cdot Thermal\ noise$, incoherent correction is proportional to the sum of particles' signals in the sample $N_s = N/(2WT_0)$ and a thermal noise

Under given assumptions, the equation for the rms-particle simplifies to

$$\frac{1}{\tau} = \frac{W}{N}[2g - g^2(1+U)],$$

where $g = \lambda N_s$, $U = Therm.noise/(x^2 N_s)$.

FOKKER-PLANCK EQUATION

Consider a following generic Langevin equation with the same statistics for $\xi(t)$ as in Equation (1):

$$\frac{dx}{dt} = a(x,t) + b(x,t)\xi(t).$$

This generic Langevin equation has a corresponding deterministic Fokker-Planck equation of the form (see e.g. [6]):

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}\left[\left(a + \frac{1}{2}b\frac{\partial b}{\partial x}\right)\Psi\right] + \frac{\partial^2}{\partial x^2}(b^2\Psi).$$

The function Ψ is a probability distribution of a single particle, but mathematically it is identical to the particle distribution function when $N \to \infty$, which could be considered true for typical beam intensities. The summand under derivative $\frac{1}{2}g\frac{\partial g}{\partial x}$ is a so-called noise-induced drift. To compensate this drift, the additional summand was added in the original Langevin Equation (1), otherwise diffusion term could lead to cooling.

Consequently, the corresponding Fokker-Planck equation for the given Langevin Equation (1) will have its traditional for stochastic cooling form:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}(F\Psi) + \frac{\partial}{\partial x}\left(D\frac{\partial \Psi}{\partial x}\right).$$

While without this compensational term there would have been a different and incorrect form of the equation:

$$\frac{\partial \Psi}{\partial t} = -\frac{\partial}{\partial x}(F\Psi) + \frac{\partial^2}{\partial x^2}(D\Psi).$$

It was quite an issue in the early days, which form of equation is suitable for the stochastic cooling, and there was provided a dedicated experiment to verify the correct form of equation [7].

For the simplified example introduced earlier, the corresponding Fokker-Planck equation could be written in the following form:

$$\frac{\partial \Psi}{\partial t} = \frac{\mathrm{W}}{\mathrm{N}}\left[2g\frac{\partial}{\partial x}(x\Psi) + g^2(1+U)\frac{\partial}{\partial x}\left(x^2\frac{\partial \Psi}{\partial x}\right)\right].$$

CONCLUSION

The stochastic cooling theory was formulated as continuous Wiener process. Such treatment derives the well-known equations in a clear and natural way, and besides it provides:

- Langevin equation for tracking simulations
- General form of equation for the single particle approach
- Explanation of the Fokker-Planck equation form
- Same well-established drift and diffusion coefficients for three theoretical approaches (incl. particle tracking).

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