MULTISCALE REPRESENTATIONS FOR SOLUTIONS OF VLASOV-MAXWELL EQUATIONS FOR INTENSE BEAM PROPAGATION

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Abstract

We present the applications of variational–wavelet approach for computing multiresolution/multiscale representation for solution of some approximations of Vlasov-Maxwell equations.

1 INTRODUCTION

In this paper we consider the applications of a new numerical-analytical technique which is based on the methods of local nonlinear harmonic analysis or wavelet analysis to the nonlinear beam/accelerator physics problems described by some forms of Vlasov-Maxwell (Poisson) equations. Such approach may be useful in all models in which it is possible and reasonable to reduce all complicated problems related with statistical distributions to the problems described by systems of nonlinear ordinary/partial differential equations. Wavelet analysis is a relatively novel set of mathematical methods, which gives us the possibility to work with well-localized bases in functional spaces and gives for the general type of operators (differential, integral, pseudodifferential) in such bases the maximum sparse forms. Our approach in this paper is based on the generalization of variational-wavelet approach from [1]-[8], which allows us to consider not only polynomial but rational type of nonlinearities [9]. The solution has the following form (related forms in part 3)

$$
\begin{aligned}
u(t,x) &= \sum_{k\in Z^n} U^k(x)V^k(t), \qquad (1)\\
V^k(t) &= V_N^{k,slow}(t) + \sum_{j\ge N} V_j^k(\omega_j^1 t), \quad \omega_j^1 \sim 2^j\\
U^k(x) &= U_N^{k,slow}(x) + \sum_{j\ge N} U_j^k(\omega_j^2 x), \quad \omega_j^2 \sim 2^j
\end{aligned}
$$

which corresponds to the full multiresolution expansion in all time/space scales.

Formula (1) gives us expansion into the slow part u_N^{slow} and fast oscillating parts for arbitrary N. So, we may move from coarse scales of resolution to the finest one for obtaining more detailed information about our dynamical process. The first term in the RHS of formulae (1) corresponds on the global level of function space decomposition to resolution space and the second one to detail space. In this way we give contribution to our full solution from each scale of resolution or each time/space scale. The same is correct for the contribution to power spectral density (energy spectrum): we can take into account contributions from each level/scale of resolution. Starting in part 2 from Vlasov-Maxwell equations we consider in part 3 the generalization of our approach based on variational formulation in the biorthogonal bases of compactly supported wavelets.

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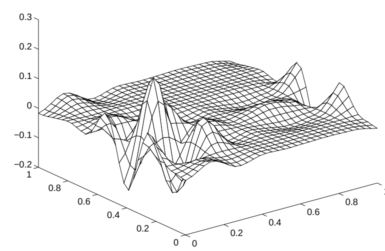

Figure 1: Base wavelet

2 VLASOV-MAXWELL EQUATIONS

Analysis based on the non-linear Vlasov-Maxwell equations leds to more clear understanding of the collective effects and nonlinear beam dynamics of high intensity beam propagation in periodic-focusing and uniform-focusing transport systems. We consider the following form of equations ([11] for setup and designation):

$$
\begin{aligned}
&\Big\{\frac{\partial}{\partial s} + x'\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} - \Big[k_x(s)x + \frac{\partial\psi}{\partial x}\Big]\frac{\partial}{\partial x'} -\\
&\Big[k_y(s)y + \frac{\partial\psi}{\partial y}\Big]\frac{\partial}{\partial y'}\Big\} f_b = 0, \qquad (2)\\
&\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\psi = -\frac{2\pi K_b}{N_b}\int \mathrm{d}x'\mathrm{d}y' f_b. \qquad (3)
\end{aligned}
$$

The corresponding Hamiltonian for transverse single-particle motion is given by

$$
\begin{aligned}
\hat H(x,y,x',y',s) &= \frac{1}{2}(x'^2 + y'^2)\\
&+\frac{1}{2}[k_x(s)x^2 + k_y(s)y^2] + \psi(x,y,s). \qquad (4)
\end{aligned}
$$

Related Vlasov system describes longitudinal dynamics of high energy stored beam [12]:

$$
\begin{aligned}
&\frac{\partial f}{\partial T} + v\frac{\partial f}{\partial \theta} + \lambda V\frac{\partial f}{\partial v} = 0, \qquad (5)\\
&\frac{\partial^2 V}{\partial T^2} + 2\gamma\frac{\partial V}{\partial T} + \omega^2 V = \frac{\partial I}{\partial T} \qquad (6)\\
&I(\theta;T) = \int \mathrm{d}v v f(\theta,v;T). \qquad (7)
\end{aligned}
$$

3 VARIATIONAL APPROACH IN BIORTHOGONAL WAVELET BASES

Now we consider some useful generalization of our variational wavelet approach. Because integrand of variational functionals is represented by bilinear form (scalar product) it seems more reasonable to consider wavelet constructions [13] which take into account all advantages of this structure. The action functional for loops in the phase space is

$$F(\gamma) = \int_\gamma pdq - \int_0^1 H(t,\gamma(t))dt \qquad (8)$$

The critical points of F are those loops γ, which solve the Hamiltonian equations associated with the Hamiltonian H and hence are periodic orbits. Let us consider the loop space $\Omega = C^\infty(S^1, R^{2n})$, where $S^1 = R/\mathbf{Z}$, of smooth loops in R^{2n}. Let us define a function $\Phi : \Omega \to R$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} < -J\dot{x}, x > dt - \int_0^1 H(x(t))dt, \quad x\in\Omega \qquad (9)$$

Computing the derivative at $x\in\Omega$ in the direction of $y\in\Omega$, we find

$$\Phi'(x)(y) = \int_0^1 < -J\dot{x} - \bigtriangledown H(x), y > dt \qquad (10)$$

Consequently, $\Phi'(x)(y) = 0$ for all $y\in\Omega$ iff the loop x is a solution of the Hamiltonian equations. Now we need to take into account underlying bilinear structure via wavelets. We started with two hierarchical sequences of approximations spaces [13]: $\dots V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\dots$, $\dots\widetilde{V}_{-2}\subset\widetilde{V}_{-1}\subset\widetilde{V}_0\subset\widetilde{V}_1\subset\widetilde{V}_2\dots$, and as usually, W_0 is complement to V_0 in V_1, but now not necessarily orthogonal complement. New orthogonality conditions have now the following form: $\widetilde{W}_0\perp V_0,\quad W_0\perp\widetilde{V}_0,\quad V_j\perp\widetilde{W}_j,\quad \widetilde{V}_j\perp W_j$ translates of ψ span W_0, translates of $\tilde\psi$ span $\widetilde{W}_0$. Biorthogonality conditions are $<\psi_{jk},\tilde\psi_{j'k'}>=\int_{-\infty}^{\infty}\psi_{jk}(x)\tilde\psi_{j'k'}(x)\mathrm{d}x = \delta_{kk'}\delta_{jj'}$, where $\psi_{jk}(x) = 2^{j/2}\psi(2^jx-k)$. Functions $\varphi(x),\tilde\varphi(x-k)$ form dual pair: $<\varphi(x-k),\tilde\varphi(x-\ell)>=\delta_{kl}$, $<\varphi(x-k),\tilde\psi(x-\ell)>=0$ for $\forall k,\ \forall\ell$. Functions $\varphi,\tilde\varphi$ generate a multiresolution analysis. $\varphi(x-k)$, $\psi(x-k)$ are synthesis functions, $\tilde\varphi(x-\ell)$, $\tilde\psi(x-\ell)$ are analysis functions. Synthesis functions are biorthogonal to analysis functions. Scaling spaces are orthogonal to dual wavelet spaces. Two multiresolutions are intertwining $V_j + W_j = V_{j+1},\quad \widetilde{V}_j+\widetilde{W}_j = \widetilde{V}_{j+1}$. These are direct sums but not orthogonal sums. So, our representation for solution has now the form

$$f(t) = \sum_{j,k}\tilde b_{jk}\psi_{jk}(t), \qquad (11)$$

where synthesis wavelets are used to synthesize the function. But $\tilde b_{jk}$ come from inner products with analysis wavelets. Biorthogonality yields

$$\tilde b_{\ell m} = \int f(t)\tilde\psi_{\ell m}(t)\mathrm{d}t. \qquad (12)$$

So, now we can introduce this more useful construction into our variational approach. We have modification only on the level of computing coefficients of reduced nonlinear algebraical system. This new construction is more flexible. Biorthogonal point of view is more stable under the action of large class of operators while orthogonal (one scale for multiresolution) is fragile, all computations are much more simpler and we accelerate the rate of convergence. In all types of (Hamiltonian) calculation, which are based on some bilinear structures (symplectic or Poissonian structures, bilinear form of integrand in variational integral) this framework leads to greater success.

So, we try to use wavelet bases with their good spatial and scale–wavenumber localization properties to explore the dynamics of coherent structures in spatially-extended, 'turbulent'/stochastic systems. After some ansatzes and reductions we arrive from (2),(3) or (5)-(7) to some system of nonlinear partial differential equations [10]. We consider application of our technique to Kuramoto-Sivashiinsky equation as a model with rich spatio-temporal behaviour [14] ($0\le x\le L,\quad \xi = x/L,\quad u(0,t) = u(L,t)$, $u_x(0,t) = u_x(L,t)$):

$$\begin{aligned} u_t &= -u_{xxx} - u_{xx} - uu_x = Au + B(u)\\ u_t &+ \frac{1}{L^4}u_{\xi\xi\xi\xi} + \frac{1}{L^2}u_{\xi\xi} + \frac{1}{L}uu_\xi = 0 \end{aligned} \qquad (13)$$

Let be

$$u(x,t) = \sum_{k=0}^N\sum_{\ell=0}^M a_\ell^k(t)\psi_\ell^k(\xi) = \sum a_\ell^k\psi_\ell^k, \qquad (14)$$

where $\psi_\ell^k(\xi)$, $a_\ell^k(t)$ are both wavelets.
Variational formulation

$$\left(\sum_{k,\ell}\Big\{\dot a_\ell^k\psi_\ell^k + \frac{1}{L^4}a_\ell^k\psi_\ell^{k''''} + \frac{1}{L^2}a_\ell^k\psi_\ell^{k''} + \frac{1}{L}\sum_{p,q}a_\ell^k a_q^p\psi_\ell^k\psi_q^{p'}\Big\},\psi_s^r\right) = 0 \qquad (15)$$

reduces (13) to ODE and algebraical one.

$$\begin{aligned} M_{s\ell}^{rk}\dot a_s^r &= \sum_{k,\ell}L_{s\ell}^{rk}a_\ell^k + \sum_{k,\ell}\sum_{p,q}N_{sq\ell}^{rpk}a_q^pa_\ell^k\\ M_{s\ell}^{rk} &= (\psi_\ell^k,\psi_s^r)\\ L_{s\ell}^{rk} &= \frac{1}{L^2}(\psi_s^{r'},\psi_\ell^{k'}) - \frac{1}{L^4}(\psi_s^{r''},\psi_\ell^{k''})\\ N_{sq\ell}^{rpk} &= \frac{1}{L}(\psi_s^r,\psi_q^p\psi_\ell^{k'}) \end{aligned} \qquad (16)$$

In particular case on $V_2\setminus V_0$ we have:

$$\begin{pmatrix}\dot a_0\\ \dot a_1\\ \dot a_2\end{pmatrix} = \Big[L\Big]\begin{pmatrix}a_0\\ a_1\\ a_2\end{pmatrix} +$$

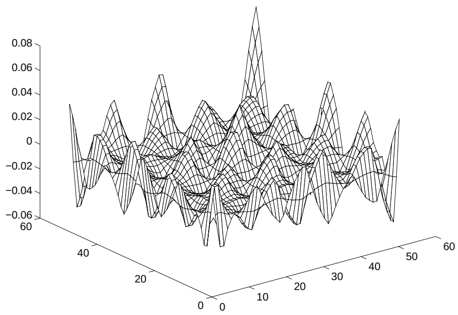

Figure 2: The solution of eq.(13)

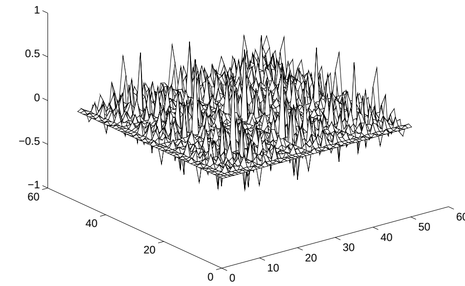

Figure 3: The solution of eq.(17)

$$\begin{pmatrix} ca_0a_1 - ca_0a_2 + da_1^2 - da_2^2 \\ -ca_0^2 - da_0a_1 + \ell a_0a_2 - fa_1a_2 - fa_2^2 \\ ca_0^2 - \ell a_0a_1 + da_0a_1 + da_0a_2 + fa_1^2 + fa_1a_2 \end{pmatrix}$$

Then in contrast to [14] we apply to (16) methods from [1]-[9] and arrive to formula (1). The same approach we use for the general nonlinear wave equation

$$u_{tt} = u_{xx} - mu - f(u), \tag{17}$$

where

$$f(u) = au^3 + \sum_{k\geq 5} f_k u^k \tag{18}$$

According to [2],[10] we may consider it as infinite dimensional Hamiltonian systems with phase space $= H_0^1 \times L^2$ on $[0, L]$ and coordinates: $u, v = u_t$, then

$$\begin{aligned} H &= \frac{1}{2} < v, v > + \frac{1}{2} < Au, u > + \int_0^\pi g(u)\mathrm{d}x \\ A &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + m, \qquad g = \int f(s)\mathrm{d}s \\ u_t &= \frac{\partial H}{\partial v} = v \qquad (19) \\ v_t &= -\frac{\partial H}{\partial u} = -Au - f(u) \\ \text{or} \quad \dot{u}(t) &= J\nabla K(u(t)) \end{aligned}$$

Then anzatzes:

$$\begin{aligned} u(t,x) &= U(\omega_1 t, \dots, \omega_n t, x) \qquad (20) \\ u(t,x) &= \sum_{k\in\mathcal{Z}^n} U_k(x)\exp(ik\cdot\omega(k)t) \\ u(t,x) &= S(x - vt) \\ u(t,x) &= \sum_{k\in\mathcal{Z}^n} U_k(x)V_k(t) \end{aligned}$$

and methods [1]-[10] led to formulae (1). Resulting multiresolution/multiscale representation in the high-localized bases (Fig.1) is demonstrated on Fig.2, Fig.3. We would like to thank Professor James B. Rosenzweig and Mrs. Melinda Laraneta for nice hospitality, help and support during UCLA ICFA Workshop.

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