

PERTURBATIVE APPROACH TO NONLINEAR CHROMATICITY OF A CIRCULAR ACCELERATOR

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Abstract

Using the expressions for the higher order terms of a nonlinear dispersion function of a ring accelerator, we derive the higher order formula for the nonlinear chromaticity. Numerical estimation of the second order nonlinear chromaticity of the SPRING-8 storage ring is carried out, which agrees fairly well with the measurement.

1 INTRODUCTION

The larger a size of a storage ring becomes, the bigger value the chromaticity reaches, which should be corrected by strong sextupole magnets. Hence, for the purpose of precisely controlling the chromaticity, it is necessary to know the nonlinear behavior. In this paper we derive the perturbative expression for the higher order terms of nonlinear chromaticity in terms of the nonlinear dispersion function, whose general formula we have derived before [1].

Calculating the nonlinear chromaticity of the SPRING-8 storage ring as an example, we prove the validity of the formula.

2 FORMULATION

2.1 Hamiltonian

As usual, we employ the curvilinear coordinate system, where the Hamiltonian describing the motion of a particle is

$$H(x, p_x, y, p_y) = -(1 + K_x x) \sqrt{(1 + \delta)^2 - p_x^2 - p_y^2} + \frac{1}{2} (1 + K_x x)^2 + \sum_{n=0} \frac{g_n}{(n+2)!} \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor + 1} (-1)^m \binom{n+2}{m} x^{n+2-2m} y^{2m},$$

where δ is the fractional deviation of the momentum $\delta = (p - p_0) / p_0$ with the nominal momentum p_0 , and K_x is the horizontal curvature, and g_n 's ($n = 0, 1, \dots$) are the strengths of quadrupole, sextupole, and so on. Here $[\cdot]$ represents the Gauss symbol and $\binom{\cdot}{\cdot}$ the binomial coefficient. In our formulation the following properties for the magnet system are assumed:

- There is no vertical curvature.
- There is no skew magnetic element.

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- All magnets are separate function type.
- All magnets are approximated to have no fringe field or to possess hard edge.

2.2 Dispersion Function

The off-momentum trajectory is described by the dispersion function $x_\epsilon(s)$ and the conjugate momentum $p_\epsilon(s)$ satisfying the following equations

$$x'_\epsilon = (1 + K_x x_\epsilon) \frac{p_\epsilon}{\sqrt{(1 + \delta)^2 - p_\epsilon^2}},$$

$$p'_\epsilon = K_x \left[\sqrt{(1 + \delta)^2 - p_\epsilon^2} - 1 \right] - K_x^2 x_\epsilon - \sum_{n=1} \frac{g_{n-1}}{n!} x_\epsilon^n,$$

which can be solved by the perturbative expansion

$$x_\epsilon(s) = \delta \sum_{n=0} \delta^n \eta_n(s), \quad p_\epsilon(s) = \delta \sum_{n=0} \delta^n \zeta_n(s).$$

Eliminating the momenta ζ_n , we derive the recurrent equations for higher order terms of the nonlinear dispersion function η_n

$$\eta_n'' + (K_x^2 + g_0) \eta_n = \Omega_n(\eta_0, \dots, \eta_{n-1}), \quad (1)$$

where explicit forms of some low order inhomogeneous terms $\Omega_n(\eta_0, \dots, \eta_{n-1})$ are $\Omega_0 = K_x$,

$$\Omega_1 = g_0 \eta_0 - \frac{g_1 \eta_0^2}{2} - K_x \left(1 - \frac{\eta_0'^2}{2} \right) + 2K_x^2 \eta_0 - K_x^3 \eta_0^2,$$

and so on. Of course, the lowest equation ($n = 0$) of Eq. (1) is the one determining the linear dispersion. In [1], by solving the recurrence equation Eq. (1), the explicit expressions of the higher order terms of the dispersion function up to 4-th are given.

2.3 Chromaticity

To calculate the chromaticity, we perform the canonical transformation which shifts the origin of phase space to (x_ϵ, p_ϵ) . The transformation $(x, p_x) \mapsto (x_\beta, p_\beta)$ can be performed with the generating function

$$F_2(x, p_\beta) = (x - x_\epsilon(s))(p_\beta + p_\epsilon(s)),$$

which yields the transformation equations

$$x = x_\beta + x_\epsilon(s), \quad p_x = p_\beta + p_\epsilon(s), \quad H_\beta = H + \frac{\partial F_2}{\partial s},$$

where the identity transformation for y , and p_y has been suppressed.

From now on, we consider the linear motion of the betatron oscillation around the off-momentum trajectory $(x_\epsilon(s), p_\epsilon(s))$. Then, the Hamiltonian H_β with respect to x_β, y, p_β and p_y up to second order is given by

$$H_\beta = \frac{1 + K_x x_\epsilon}{2\sqrt{(1+\delta)^2 - p_\epsilon^2}} \left[\frac{(1+\delta)^2}{(1+\delta)^2 - p_\epsilon^2} p_\beta^2 + p_y^2 \right] + \frac{K_x p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} x_\beta p_\beta + \frac{1}{2} K_x^2 x_\beta^2 + \frac{1}{2} \left(\sum_{n=0} g_n x_\epsilon^n \right) (x_\beta^2 - y^2).$$

The equations of motion obtained from the above Hamiltonian are

$$\frac{d}{ds} \begin{pmatrix} x_\beta \\ p_\beta \end{pmatrix} = \mathbf{T}_x(s) \begin{pmatrix} x_\beta \\ p_\beta \end{pmatrix}, \quad \frac{d}{ds} \begin{pmatrix} y \\ p_y \end{pmatrix} = \mathbf{T}_y(s) \begin{pmatrix} y \\ p_y \end{pmatrix},$$

where \mathbf{T}_z 's ($z = x, y$) are the instant transformation matrices:

$$\mathbf{T}_x(s) = \begin{pmatrix} \frac{K_x p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} & \frac{(1+\delta)^2 (1 + K_x x_\epsilon)}{[(1+\delta)^2 - p_\epsilon^2]^{3/2}} \\ -K_x^2 - \sum_{n=0} \frac{g_n}{n!} x_\epsilon^n & -\frac{K_x p_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} \end{pmatrix} \equiv \begin{pmatrix} A_x(s) & B_x(s) \\ -C_x(s) & -A_x(s) \end{pmatrix},$$

$$\mathbf{T}_y(s) = \begin{pmatrix} 0 & \frac{1 + K_x x_\epsilon}{\sqrt{(1+\delta)^2 - p_\epsilon^2}} \\ \sum_{n=0} \frac{g_n}{n!} x_\epsilon^n & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & B_y(s) \\ -C_y(s) & 0 \end{pmatrix}.$$

After performing the local variable transformation

$$\begin{pmatrix} z \\ p_z \end{pmatrix} = \mathbf{U}_w(s) \begin{pmatrix} w \\ p_w \end{pmatrix}, \quad \mathbf{U}_w(s) = \begin{pmatrix} B_z^{1/2} & 0 \\ B_z^{-1/2} \left(\frac{1}{2} B_z^{-1} B'_z - A_z \right) & B_z^{-1/2} \end{pmatrix},$$

for $z = x, y$ and $w = u, v$ respectively, we have the ‘‘Hill’s equation’’

$$\frac{d}{ds} \begin{pmatrix} w \\ p_w \end{pmatrix} = \mathbf{T}_w(s) \begin{pmatrix} w \\ p_w \end{pmatrix},$$

with

$$\mathbf{T}_w(s) = \begin{pmatrix} 0 & 1 \\ -G_w(s) & 0 \end{pmatrix},$$

$$G_w(s) = B_z C_z - A'_z + \frac{(\log B_z)''}{2} - \left[A_z - \frac{(\log B_z)'}{2} \right]^2.$$

According to the standard description of the Hill’s equation, we can construct the transform matrix $\mathbf{M}_w(s_1|s_0)$

$$\begin{pmatrix} w \\ p_w \end{pmatrix} \Big|_{s_1} = \mathbf{M}_w(s_1|s_0) \begin{pmatrix} w \\ p_w \end{pmatrix} \Big|_{s_0}.$$

From the transfer matrices \mathbf{M}_w for $w = u, v$, one can easily deduce those for $z = x, y$ as

$$\mathbf{M}_z(s_1|s_0) = \mathbf{U}_w(s_1) \mathbf{M}_w(s_1|s_0) \mathbf{U}_w^{-1}(s_0). \quad (2)$$

As usual one can relate the tune $\nu_w \equiv \mu_w / (2\pi)$ to the transfer matrix as

$$\cos \mu_w = \cos 2\pi \nu_w = \frac{1}{2} \text{Tr} \mathbf{M}_w(s_0 + L|s_0). \quad (3)$$

Owing to the relation (2) and the periodicity of the lattice functions, we can find that the tunes ν_z for the x - y system to be equal to those in the u - v system.

Now, we perturbatively calculate the nonlinear chromaticity based on the defining equation (3). From now on, for the simplicity, we omit the suffix denoting the coordinates such as x, y or u, v if not necessary. Expanding the phase advance μ throughout the circumference with respect to the momentum deviation δ

$$\mu = \sum_{n=0} \delta^n \mu_n,$$

we have

$$\cos \mu = \sum_{n=0} \delta^n \chi_n,$$

where

$$\chi_0 = \cos \mu_0, \quad \chi_1 = -\mu_1 \sin \mu_0, \quad \chi_2 = -\mu_2 \sin \mu_0 - \frac{1}{2} \mu_1^2 \cos \mu_0,$$

and so on.

On the other hand, since

$$\sum_{n=0} \delta^n \chi_n = \frac{1}{2} \text{Tr} \mathbf{M}(s_0 + L|s_0),$$

the expansion of the instant transfer matrix \mathbf{T} with respect to δ

$$\mathbf{T}(s) = \sum_{n=0} \delta^n \mathbf{T}_n(s)$$

gives

$$\chi_0 = \frac{1}{2} \text{Tr} \mathbf{M}_0(s_0 + L|s_0), \quad \chi_1 = \frac{1}{2} \int_{s_0}^{s_0+L} ds_1 \text{Tr} [\mathbf{M}_0(s_1 + L|s_1) \mathbf{T}_1(s_1)], \quad \chi_2 = \frac{1}{2} \int_{s_0}^{s_0+L} ds_1 \text{Tr} [\mathbf{M}_0(s_1 + L|s_1) \mathbf{T}_2(s_1)] + \frac{1}{2} \int_{s_0}^{s_0+L} ds_2 \int_{s_0}^{s_2} ds_1 \text{Tr} [\mathbf{M}_0(s_1 + L|s_2) \times \mathbf{T}_1(s_2) \mathbf{M}_0(s_2|s_1) \mathbf{T}_1(s_1)],$$

and so on.

The zeroth order equation merely defines the tune of the central energy ($\delta = 0$) (see Eq. (3)). As easily found

$$\mu_1 = \frac{1}{2} \int_{s_0}^{s_0+L} ds_1 \beta(s_1) G_1(s_1),$$

the first order equation is the well-known definition of the linear chromaticity: *e.g.* for horizontal

$$\xi_{1x} \left(\equiv \frac{\mu_{1x}}{2\pi} \right) = \frac{1}{4\pi} \int_{s_0}^{s_0+L} ds_1 \left[-\beta_x (K_x^2 + g_0 - g_1 \eta_0) - 2\alpha_x K_x \eta_0' + \gamma_x K_x \eta_0 \right],$$

where we have performed the partial integration. After lengthy calculation, the second order nonlinear chromaticity $\xi_2 \equiv \mu_2/(2\pi)$ is given by

$$\xi_2 = \frac{1}{4\pi} \int_{s_0}^{s_0+L} ds_1 \beta(s_1) G_2(s_1) - \frac{1}{64\pi} \mu_0 a_1^2(0) - \sum_{n=1}^{\infty} \frac{\mu_0^3}{32\pi(\mu_0^2 - \pi^2 n^2)} [a_1^2(n) + b_1^2(n)]$$

with the Fourier components of $\beta(s)G_1(s)$

$$a_1(n) = \frac{2}{\mu_0} \int_{s_0}^{s_0+L} ds \cos \left[\frac{2\pi n \varphi(s)}{\mu_0} \right] \beta(s) G_1(s),$$

$$b_1(n) = \frac{2}{\mu_0} \int_{s_0}^{s_0+L} ds \sin \left[\frac{2\pi n \varphi(s)}{\mu_0} \right] \beta(s) G_1(s)$$

for ($n = 1, 2, \dots$). Here $\varphi(s) = \int_{s_0}^s ds_1/\beta(s_1)$. The explicit forms of G_2 , a_1 and b_1 as well as the third order formula are given elsewhere [2].

3 NUMERICAL CALCULATION

Applying our formula of the nonlinear chromaticity to the SPring-8 storage ring and comparing the result with the measurement, we confirm the validity of the formulation. The lattice of the SPring-8 storage ring is double bend achromat, *i.e.* Chasman-Green lattice, composed of 48 unit cells. Four of 48 cells lack bending magnets for the purpose of installing the long straight sections of 30 m. To cancel the chromatic effect, the strong sextupole magnets are installed, so that the nonlinearity of the optics is considerably strong.

We have three sets of the chromaticity measurement data, which were performed at the different operation points $(\nu_x, \nu_y) = (43.16, 21.36)$, $(43.31, 21.36)$ and $(43.46, 21.36)$. In all the cases the linear chromaticities are set to $(\xi_x, \xi_y) = (7, 4)$.

In Figure 1, the measured horizontal tunes are represented by the solid circles, and the vertical one by the open ones, respectively. The error of the tune measurement, mainly coming from the fluctuation of the tune itself, is less than 0.005. To convert the rf frequency into the momentum deviation, the momentum compaction factor calculated by

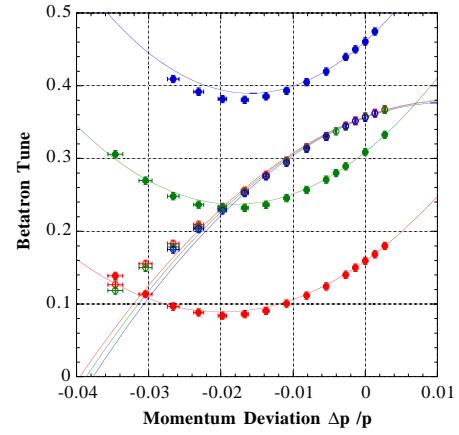


Figure 1: The measured tunes as a function of momentum deviation. The red circles (triangle) correspond to the measured horizontal (vertical) tunes of operation point with central energy (46.16, 21.36). The green and the blue symbols indicate the ones of (43.31, 21.36) and (43.46, 21.36), respectively. The corresponding calculated results are represented by the solid lines.

means of the nonlinear dispersion, is used, so that the momentum deviation estimation error is presumed to be less than 3%.

In the numerical calculation, the summation of the Fourier components is performed with confirming the convergence. Although the convergence is poor for a large machine like the SPring-8 storage ring, even the second order calculation shows the exceedingly well agreement with the measurement.

For the larger momentum deviation the discrepancy between the measurement and the calculation gradually grows up. It suggests the effect of the higher order terms than the second.

4 SUMMARY

We derived the exact formula for the second order chromaticity, whose efficiency is proved by the application to the SPring-8 storage ring. In order to improve the approximation, one should take the next order effect into account. The third order formula will be given in [2].

REFERENCES

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