TRANSVERSE COHERENT INSTABILITIES IN THE PRESENCE OF LINEAR COUPLING

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Abstract

Strong coupling between the transverse planes of a particle beam leads to an "equipartition" of the oscillation energy, including the growth rates in the case of coherent instability. The aim of this paper is to give a general formula, which includes linear coupling and which extends to two dimensions the one-dimensional results of transverse coherent instabilities. From this equation, previous results are recovered as expected: (i) Sacherer's formula for the coherent modes of oscillation, (ii) Kohaupt's formula for the mode coupling instability, and (iii) the coupled Landau damping mechanism (transfer of frequency spread), which includes the sharing of the instability growth rates. Measurements have been performed in the CERN PS, which confirm the predicted beneficial effect of coupling by both frequency spread and chromaticity sharing.

1 INTRODUCTION

Two mechanisms are widely used to damp transverse coherent instabilities. The first one is Landau damping through non-linearities, which induce a spread in the betatron frequencies via the dependence on the incoherent betatron amplitudes. If the coherent betatron frequency lies within this spread, then the motion can lose its coherence, and the beam is stabilised. However, too strong non-linearities are harmful because they can create stop-bands. The second method consists in using an electronic feedback system, which detects and counteracts the coherent motion. It is shown in this paper that linear coupling with skew quadrupoles is an additional (3rd) method that can be used to damp the transverse coherent motion, especially if the instability tends to occur only in one of the planes.

2 THEORY

2.1 A General Formula for the Transverse Coherent Instabilities

In the presence of linear coupling (near the coupling resonance $Q_X - Q_Y = l$), the stability of intense beams can be discussed using the following determinant [1]

$$\begin{vmatrix} I_{x,m}^{-1} - \Delta\omega_{m,m}^{x} & -\Delta\omega_{m,m+1}^{x} & -\dfrac{\hat{K}_0(l)\,R^2\,\Omega_0^2}{2\,\omega_{x0}} & 0 \\ \Delta\omega_{m,m+1}^{x} & I_{x,m+1}^{-1} - \Delta\omega_{m+1,m+1}^{x} & 0 & -\dfrac{\hat{K}_0(l)\,R^2\,\Omega_0^2}{2\,\omega_{x0}} \\ -\dfrac{\hat{K}_0(-l)\,R^2\,\Omega_0^2}{2\,\omega_{y0}} & 0 & I_{y,m}^{-1} - \Delta\omega_{m,m}^{y} & -\Delta\omega_{m,m+1}^{y} \\ 0 & -\dfrac{\hat{K}_0(-l)\,R^2\,\Omega_0^2}{2\,\omega_{y0}} & \Delta\omega_{m,m+1}^{y} & I_{y,m+1}^{-1} - \Delta\omega_{m+1,m+1}^{y} \end{vmatrix} = 0, \quad (1)$$

with

$$I_{x,m} = \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{y}=0}^{\hat{y}=+\infty} \frac{-2\pi^2 \dfrac{df_{x0}(\hat{x})}{d\hat{x}}\,\hat{x}^2\, f_{y0}(\hat{y})\,\hat{y}}{\omega_c - \omega_x(\hat{x},\hat{y}) - m\,\omega_s}\, d\hat{x}\, d\hat{y}, \quad (2)$$

$$I_{y,m} = \int_{\hat{x}=0}^{\hat{x}=+\infty} \int_{\hat{y}=0}^{\hat{y}=+\infty} \frac{-2\pi^2 \dfrac{df_{y0}(\hat{y})}{d\hat{y}}\,\hat{y}^2\, f_{x0}(\hat{x})\,\hat{x}}{\omega_c - \omega_y(\hat{x},\hat{y}) - l\,\Omega_0 - m\,\omega_s}\, d\hat{x}\, d\hat{y}, \quad (3)$$

$$\Delta\omega_{m,m}^{x,y} = \left(|m|+1\right)^{-1} \frac{j\,e\,\beta\,I_b}{2\,m_0\,\gamma\,Q_{x0,y0}\,\Omega_0\,L} \left(Z_{x,y}^{eff}\right)_{m,m}, \quad (4)$$

$$\Delta\omega_{m,m+1}^{x,y} = \left(|m|+1\right)^{-1} \frac{j\,e\,\beta\,I_b}{2\,m_0\,\gamma\,Q_{x0,y0}\,\Omega_0\,L} \left(Z_{x,y}^{eff}\right)_{m,m+1}, \quad (5)$$

$$\left(Z_{x,y}^{eff}\right)_{m,n} = \frac{\sum_{k=-\infty}^{k=+\infty} Z_{x,y}\left(\omega_k^{x,y}\right) h_{m,n}\left(\omega_k^{x,y} - \omega_{\xi_{x,y}}\right)}{\sum_{k=-\infty}^{k=+\infty} h_{m,m}\left(\omega_k^{x,y} - \omega_{\xi_{x,y}}\right)}, \quad (6)$$

$$h_{m,m}(\omega) = \frac{\tau_b^2}{2\pi^4}\left(|m|+1\right)^2 \times \left\{1 + (-1)^{|m|} \cos\left[\omega\,\tau_b\right]\right\} \times \left\{\left(\omega\,\tau_b/\pi\right)^2 - \left(|m|+1\right)^2\right\}^{-2}, \quad (7)$$

$$h_{m,m+1}(\omega) = \frac{\tau_b^2}{2\pi^4\,j} \sin\left[\omega\,\tau_b\right] \times \left(|m|+1\right)\left(|m|+2\right) \times \left\{\left(\omega\,\tau_b/\pi\right)^2 - \left(|m|+1\right)^2\right\}^{-1} \left\{\left(\omega\,\tau_b/\pi\right)^2 - \left(|m|+2\right)^2\right\}^{-1}. \quad (8)$$

Here, $I_{x,m}$ and $I_{y,m}$ are the horizontal and vertical dispersion integrals, ω_c is the coherent frequency to be determined, $\omega_x(\hat{x},\hat{y})$ and $\omega_y(\hat{x},\hat{y})$ are the transverse incoherent betatron frequencies of the particles, $f_{x0}(\hat{x})$

and $f_{y0}(\hat{y})$ are the uncorrelated distribution functions of the incoherent betatron amplitudes, ω_s is the synchrotron frequency and $m = ..., -1, 0, 1, ...$ is the head-tail mode number. Furthermore, $\hat{\underline{K}}_0(l)$ is the lth Fourier coefficient of the skew gradient $\underline{K}_0 = (e / p_0)(\partial B_x / \partial x)$, with e the elementary charge, p_0 the design momentum and B_x the horizontal magnetic field, R is the average radius of the machine, Ω_0 is the average revolution frequency of the particles, $\omega_{x0,y0} = Q_{x0,y0}\,\Omega_0$ are the unperturbed betatron frequencies, $\Delta\omega_{m,m}^{x,y}$ are the complex betatron frequency shifts given by the Sacherer's formula (4) [2], $j = \sqrt{-1}$ is the imaginary unit, β and γ are the relativistic velocity and mass factors, $I_b = N_b e \Omega_0 / (2\pi)$ is the current in one bunch, m_0 is the proton rest mass, L is the total bunch length (in metres), $Z_{x,y}$ are the coupling impedances, $\omega_k^{x,y} = (k + Q_{x0,y0})\Omega_0 + m\omega_s$ with $-\infty \le k \le +\infty$, $\omega_{\xi_{x,y}} = (\xi_{x,y}/\eta) Q_{x0,y0}\Omega_0$ are the transverse chromatic frequencies, with $\xi_{x,y} = (d\,Q_{x,y} / d\,p)(p_0 / Q_{x0,y0})$ the chromaticities, and $\eta = \gamma_{tr}^{-2} - \gamma^{-2}$ is the slippage factor.

2.2 Situation in the Absence of Linear Coupling

In the absence of linear coupling, the determinant of Eq. (1) is the product of the two one-dimensional determinants. It is known from the one-dimensional theory that both determinants are then equal to zero. Below the mode coupling threshold, the coupling terms between modes m and m+1 are neglected, and the following dispersion equations for each mode m are obtained, $I_{x,m}^{-1} = \Delta\omega_{m,m}^{x}$ and $I_{y,m}^{-1} = \Delta\omega_{m,m}^{y}$. In the absence of Landau damping, the stability condition for the mth mode is $\mathrm{Im}(\Delta\omega_{m,m}^{x,y}) \ge 0$, where $\mathrm{Im}(\,)$ stands for imaginary part [2]. In the presence of Landau damping, a simplified stability criterion, which is drawn from dispersion relation analysis considering "elliptical" betatron frequency distributions, is $\Delta\omega_{x,y} \ge 2\,|\Delta\omega_{m,m}^{x,y}|$, where $\Delta\omega_{x,y}$ are the half widths at the bottom of the spectra [3]. The stability criterion against the transverse mode coupling instability, neglecting Landau damping, is given by $|\Delta\omega_{m,m+1}^{x,y}| \le |\omega_s + \Delta\omega_{m+1,m+1}^{x,y} - \Delta\omega_{m,m}^{x,y}| / 2$ [4,5].

2.3 Situation in the Absence of Mode Coupling

In the presence of both linear coupling and frequency spreads, but neglecting the mode coupling terms, the following equation is obtained for each mode m [6,7]

$$\left(I_{x,m}^{-1} - \Delta\omega_{m,m}^{x}\right)\left(I_{y,m}^{-1} - \Delta\omega_{m,m}^{y}\right) = \frac{\left|\hat{\underline{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{4\,\omega_{x0}\,\omega_{y0}}. \quad (9)$$

In the absence of external non-linearities, the necessary condition for the stability of the mth mode is

$$V_{\mathrm{eq}x}^{m} + V_{\mathrm{eq}y}^{m} \le 0, \quad (10)$$

where $V_{\mathrm{eq}x,y}^{m} = -\mathrm{Im}(\Delta\omega_{m,m}^{x,y})$. If Eq. (10) is true, then it is possible to stabilise this mode by increasing the skew gradient and/or by working closer to the coupling resonance $Q_X - Q_Y = l$. The stabilising values of the modulus of the lth Fourier coefficient of the skew gradient are given by

$$\left|\hat{\underline{K}}_0(l)\right| \ge \frac{2\left[-Q_{x0}\,Q_{y0}\,V_{\mathrm{eq}x}^{m}\,V_{\mathrm{eq}y}^{m}\right]^{1/2}}{R^2\,\Omega_0} \times \frac{\left[\left(V_{\mathrm{eq}x}^{m} + V_{\mathrm{eq}y}^{m}\right)^2 + \Omega_0^2\left(Q_X - Q_Y - l\right)^2\right]^{1/2}}{-\left(V_{\mathrm{eq}x}^{m} + V_{\mathrm{eq}y}^{m}\right)}, \quad (11)$$

where $Q_{X,Y} = (\omega_{x0,y0} + U_{\mathrm{eq}x,y}^{m}) / \Omega_0$ are the horizontal and vertical coherent tunes in the presence of wake fields ($U_{\mathrm{eq}x,y}^{m} = \mathrm{Re}(\Delta\omega_{m,m}^{x,y})$, where $\mathrm{Re}(\,)$ stands for real part), but in the absence of coupling. Furthermore, in the case of coupled-bunch instabilities of M bunches, $k = n_{x,y} + k' M$ in Eq. (6) with $-\infty \le k' \le +\infty$, and the mode numbers are related by $n_x = n_y - l$.

In the presence of external non-linearities, in addition to the transfer of instability damping, there can also be a partition of Landau damping. This has been assessed in Refs. [6] and [7], using two typical frequency distributions, Lorentzian and "elliptical", knowing that they are limiting cases, modeling spectra with and without important tails, respectively. In the case of Lorentzian distributions, the necessary condition for the stability of the mth mode and the stability criterion are given by Eqs. (10) and (11), replacing $V_{\mathrm{eq}x,y}^{m}$ by $V_{\mathrm{eq}x,y}^{m} - \delta\omega_{x,y}$, where $\delta\omega_{x,y}$ are the half widths at half maximum of the spectra. In the case of elliptical distributions, the situation is more involved. If Q_X is "far" from $Q_Y + l$, then the necessary condition for the stability of the mth mode and the stability criterion are given by Eqs. (10) and (11). There is no transfer of Landau damping since the coherent tunes are too far from each other to share the stabilising spreads. If Q_X is "near" $Q_Y + l$, then in addition to the sharing of the instability growth rates, there is also a transfer of Landau damping for "optimum" coupling. The necessary condition for stability is

$$\mathrm{Re}\left[\sqrt{\Delta\omega_x^2 - \left(2U_{\mathrm{eq}x}^{m}\right)^2} + \sqrt{\Delta\omega_y^2 - \left(2U_{\mathrm{eq}y}^{m}\right)^2}\,\right] \ge 2\left(V_{\mathrm{eq}x}^{m} + V_{\mathrm{eq}y}^{m}\right). \quad (12)$$

If Eq. (12) is fulfilled, then the stabilising values of the coupling coefficient may be approximated by

$$\left|\underline{\hat{K}}_0(l)\right| \approx \frac{\sqrt{Q_{x0}\,Q_{y0}}}{R^2\,\Omega_0} \times \left\{ \begin{array}{l} -\left[\mathrm{Re}\left(\sqrt{\Delta\omega_x^2-\left(2U_{\mathrm{eq}x}^m\right)^2}\right)-2V_{\mathrm{eq}x}^m\right]\times \\ \left[\mathrm{Re}\left(\sqrt{\Delta\omega_y^2-\left(2U_{\mathrm{eq}y}^m\right)^2}\right)-2V_{\mathrm{eq}y}^m\right] \end{array} \right\}^{1/2} . \quad (13)$$

Notice that too strong a coupling is detrimental for the coupled Landau damping mechanism, since it shifts the coherent tunes outside the spectra and thus prevents Landau damping.

Notice also that in the absence of instabilities, the difference from the tunes of the two normal modes $Q_{+,-}$ is given by

$$Q_+ - Q_- = \sqrt{\left(Q_X - Q_Y - l\right)^2 + \frac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4}{Q_{x0}\,Q_{y0}}} \; . \quad (14)$$

This is the known result of linear coupling in the simplified case of the smooth approximation [8].

2.4 Situation in the Presence of both Linear Coupling and Mode Coupling

Taking into account both mode coupling and linear coupling, but neglecting Landau damping, Eq. (1) leads to a fourth-order equation, which can be written in the following form, with $\omega_{x,m} = \omega_{x0} + m\,\omega_s + \Delta\omega_{m,m}^x$ and $\omega_{y,m} = \omega_{y0} + l\,\Omega_0 + m\,\omega_s + \Delta\omega_{m,m}^y$,

$$\begin{array}{l} \left[\left(\omega_c-\omega_{x,m}\right)\left(\omega_c-\omega_{x,m+1}\right)+\left(\Delta\omega_{m,m+1}^x\right)^2\right]\times \\ \left[\left(\omega_c-\omega_{y,m}\right)\left(\omega_c-\omega_{y,m+1}\right)+\left(\Delta\omega_{m,m+1}^y\right)^2\right] = \dfrac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{4\,\omega_{x0}\,\omega_{y0}} \times \\ \left[\begin{array}{l} \left(\omega_c-\omega_{x,m}\right)\left(\omega_c-\omega_{y,m}\right)+\left(\omega_c-\omega_{x,m+1}\right)\left(\omega_c-\omega_{y,m+1}\right) \\ -\dfrac{\left|\underline{\hat{K}}_0(l)\right|^2 R^4\,\Omega_0^4}{4\,\omega_{x0}\,\omega_{y0}} - 2\,\Delta\omega_{m,m+1}^x\,\Delta\omega_{m,m+1}^y \end{array}\right] . \end{array} \quad (15)$$

The necessary condition for stability is found to be

$$\begin{array}{l} \left|\Delta\omega_{m,m+1}^x + \Delta\omega_{m,m+1}^y\right| \le \\ \dfrac{1}{2}\left|2\,\omega_s + \Delta\omega_{m+1,m+1}^x + \Delta\omega_{m+1,m+1}^y - \Delta\omega_{m,m}^x - \Delta\omega_{m,m}^y\right| . \end{array} \quad (16)$$

If Eq. (16) is fulfilled, then it is possible to stabilise the beam by linear coupling. Beam stability is obtained above a certain threshold for the coupling strength, whose value is found from Eq. (15) by increasing the coupling coefficient until the imaginary parts of the roots disappear.

3 EXPERIMENTS

Two series of measurement have been performed in the PS to verify the results predicted by this theory. The first one was made with a high-intensity bunched proton beam to illustrate the transfer of Landau damping. The predicted beneficial effect of coupling on Landau damping has been confirmed, and a factor 7 in the octupole current has been gained, which reduces the harmful external non-linearities [9]. The second experiment was made using the PS beam for the future LHC to illustrate the transfer of the instability growth rates (which, below the mode coupling threshold, depend critically on the chromaticities). The conclusion of this experiment is that the PS beam for LHC can be stabilised by linear coupling only [10].

4 CONCLUSION

The theory of transverse coherent instabilities has been extended to include linear coupling. Two beneficial effects have been emphasised: a transfer of the instability growth rates and a transfer of the stabilising frequency spreads. These results have been verified experimentally in the PS. They also explain why many high-intensity accelerators and colliders work best close to a coupling resonance $Q_X - Q_Y = l$. They can be used to find optimum values for the transverse tunes, the skew quadrupole and octupole currents, and the chromaticities.

REFERENCES

[1] R. Cappi, E. Métral, “Transverse Mode Coupling Instability in the Presence of Linear Coupling”, CERN/PS 99-063 (CA), 1999.

[2] F.J. Sacherer, Proc. 9th Int. Conf. on High Energy Acc., Stanford 1974, p. 347.

[3] A.W. Chao, *Physics of Collective Beam Instabilities in High Energy Accelerators*, New York: Wiley, 1993.

[4] R.D. Kohaupt, “Head Tail Turbulence and the Transverse Petra Instability”, DESY 80/22, 1980.

[5] B. Zotter, “Transverse Mode-Coupling and Head-Tail Turbulence”, CERN/ISR-TH/82-10, 1982.

[6] E. Métral, “Theory of Coupled Landau Damping”, Part. Accelerators, **62**(3-4), p. 259, 1999.

[7] E. Métral, Thesis, Grenoble (France), 1999.

[8] G. Guignard, “The General Theory of All Sum and Difference Resonances in a Three-Dimensional Magnetic Field in a Synchrotron”, CERN 76-06, 1976.

[9] E. Métral, “Measurements of Coupled Landau Damping of the PS Coherent Resistive Instability, CERN/PS/CA/Note 98-16, 1998.

[10] R. Cappi, R. Garoby and E. Métral, “Collective Effects in the CERN-PS Beam for LHC”, CERN/PS 99-049 (CA), 1999.