# CHROMATIC COUPLING IN LHC AND ITS CORRECTION

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## Abstract

The geometrical optical aberrations dominate the dynamic aperture in LHC and were therefore much studied. It turned out however that a large second order chromaticity was observed for some possible configurations of the field errors in the dipoles. It is shown here to be explained by a momentum dependent betatron coupling excited by the skew sextupolar component of the dipole field. This coupling is tune dependent and increases to un-acceptable values when the LHC working point is moved towards the diagonal at a position which is operational in existing machines. The numerical results are exactly predicted using canonical perturbation theory. This knowledge is used to design a simple and optimal correction system which consists in tilting some focusing sextupoles in the arcs.

## 1 INTRODUCTION

In LHC, the dominant source of skew sextupolar perturbation ($a_3$) arises in the super-conducting dipoles. Although $a_3$ vanishes by design, a non-vanishing average value is expected on each dipole production line, compatible with the manufacturing tolerances. Its value is in the range $\pm 0.87 \times 10^{-4}$ at the reference radius $R_r = 17$ mm. Being of **geometrical** nature, $a_3$ is constant with energy. The installation strategy assumes that each of the 8 LHC arcs will be equipped with dipoles from the same production line. This $a_3$ imperfection will therefore appear as a systematic per arc. The rms variation from magnet to magnet is smaller than the systematic component. Its effect, further decreased due to the very large number of dipoles (1232), is neglected in this study. The maximum integrated strength of $a_3$ in one arc is comparable to that of the lattice sextupoles, which correct a natural chromaticity of some 80 units. Although the consequences on the beam dynamics are different, it can be inferred that the systematic $a_3$ is a significant perturbation likely to require correction.

## 2 CHROMATIC COUPLING

In the presence of skew sextupolar field errors $K_2^- \equiv 2B_0/(B\rho)\,a_3/R_r^2$, a particle sees a momentum-dependent skew quadrupolar field $K_1^-$ given by

$$K_1^-(s) = K_2^-(s)\,D_x(s)\,\delta/(1+\delta) \sim K_2^-(s)\,D_x(s)\,\delta\,,$$

where $D_x(s)$ denotes the horizontal dispersion at the location $s$. This perturbation excites mainly the (1,-1) resonance. The first-order resonance theory (see e.g. [4]) yields the perturbation of the fractional betatron eigentunes $Q_{I,II}$:

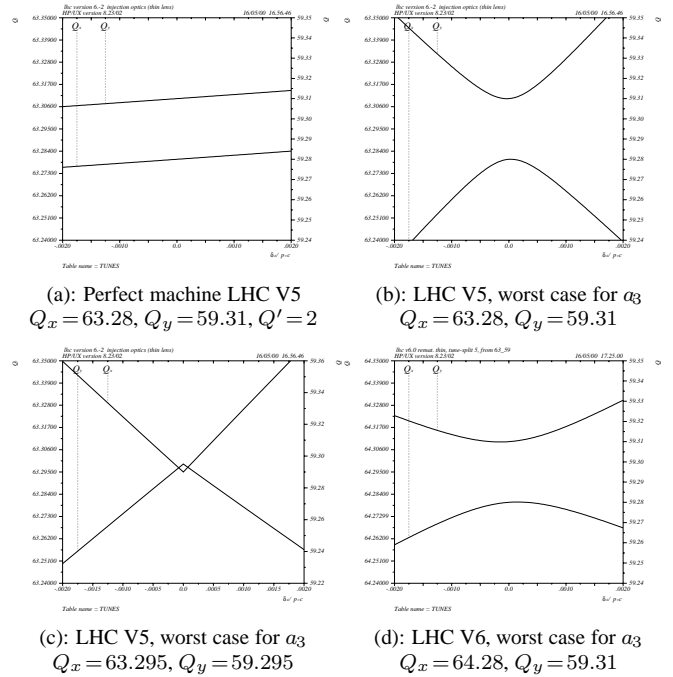$$|Q_I(\delta) - Q_{II}(\delta)| \sim \sqrt{\Delta_-^2 + |c_-|^2\,\delta^2}\,, \qquad (1)$$



(a): Perfect machine LHC V5
$Q_x = 63.28$, $Q_y = 59.31$, $Q' = 2$

(b): LHC V5, worst case for $a_3$
$Q_x = 63.28$, $Q_y = 59.31$

(c): LHC V5, worst case for $a_3$
$Q_x = 63.295$, $Q_y = 59.295$

(d): LHC V6, worst case for $a_3$
$Q_x = 64.28$, $Q_y = 59.31$

Figure 1: Tunes versus energy at injection ($-2\,10^{-3} < \delta < 2\,10^{-3}$) for LHC Versions 5 and 6

$$\text{with} \quad \begin{cases} \Delta_\pm & \stackrel{\text{def}}{=} Q_x \pm Q_y \\ c_\pm & \stackrel{\text{def}}{=} \dfrac{1}{2\pi} \displaystyle\int_0^C ds\, f(s)\, e^{i\,\mu^\pm(s)} \\ f(s) & \stackrel{\text{def}}{=} \sqrt{\beta_x(s)\beta_y(s)}\, D_x(s)\, K_2^-(s) \\ \mu^\pm(s) & \stackrel{\text{def}}{=} \mu_x(s) \pm \mu_y(s)\,. \end{cases} \qquad (2)$$

As a result, depending on $|\delta| \ll$ or $\gg |\Delta_-/c_-|$ we observe either a second- or first-order dependence of the eigentunes on momentum, respectively

$$\begin{aligned} &|Q_I(\delta) - Q_{II}(\delta)| \sim |\Delta_-| + \frac{1}{2}\,|c_-|^2/|\Delta_-|\,\delta^2\,, \text{or} \\ &|Q_I(\delta) - Q_{II}(\delta)| \sim |c_-|\,|\delta|\,. \end{aligned} \qquad (3)$$

For $\Delta_- \equiv 0$ (working point on the diagonal), note that the linear chromaticity is singular around $\delta = 0$ (see Fig. 1-c). Figure 1 shows that this perturbation is liable to produce a negative chromaticity when changing the beam momentum by small amounts, e.g. to measure chromaticity or dispersion. A head-tail instability may then be triggered. For on-momentum beams, the tunes would be modulated at twice the synchrotron frequency. Such a modulation is known to enhance the diffusion of particles in a non-linear regime.

## 3 SECOND-ORDER CHROMATICITY

Outside the sum and difference resonances $(1, \pm 1)$, the canonical perturbation theory can be used [1] to calculate exactly the (on-momentum) second-order chromaticity

due to $a_3$:

$$Q''_{I,II}(0) = \underbrace{\frac{\pi}{2}\left[\pm\cot\left[\pi\Delta_-\right]|c_-|^2 - \cot\left[\pi\Delta_+\right]|c_+|^2\right]}_{\text{Resonant terms}} -$$

$$\underbrace{\pi\left[\pm d_- - d_+\right]}_{\text{Non-resonant terms}}, \text{ with}$$

$$d_\pm \stackrel{\text{def}}{=} \frac{1}{4\pi^2}\int_0^C ds \int_0^s ds' f(s)f(s')\,\sin\left[\mu^\pm(s') - \mu^\pm(s)\right]. \quad (4)$$

A simple but rather faithful model of the LHC optics may be used to calculate analytically the respective contributions of the difference and sum resonances and of the non-resonant terms. An LHC ring is made of $N = 8$ arcs, each arc containing $N_c = 23$ identical FODO cells of betatron phase advances $\mu_x$ and $\mu_y$. $a_3$ is constant in each arc and vanishes in the insertions which simply act as phase trombones. The coefficients $c_\pm$ can thus be constructed from the contribution of one single cell $c_\pm^{(cell)}$ and from phasors (see [1] for more details):

$$c_\pm = c_\pm^{(cell)} \times \underbrace{\frac{\sin\left(N_c(\mu_x\pm\mu_y)/2\right)}{\sin\left((\mu_x\pm\mu_y)/2\right)}}_{\substack{\text{Form factor } f_\pm \text{ coming} \\ \text{from the arc}}} \times \underbrace{\sum_{k=1}^N e^{i(\mu_{x_k}\pm\mu_{y_k})}(K_2^-L)_k}_{\text{Form factor } \mathbf{F}_\pm} (5)$$

where $\mu_k$ denotes the phase advances (hor. or vert.) at the middle of the $k^{\text{th}}$ arc and $(K_2^-L)_k$ is the integrated skew sextupolar strength *per dipole* in arc $k$. The coefficients

$$c_\pm^{(cell)} \stackrel{\text{def}}{=} \int_{-L_c}^{L_c} ds \sqrt{\beta_x\beta_y} D_x e^{i(\mu_x\pm\mu_y)} \; (L_c, \text{ half-cell length})$$

depends only on the arc cell optics. For $\mu_x \sim \mu_y \sim \mu$, their expressions become [1]:

$$c_-^{(cell)} = \frac{\alpha L_c^2}{4\pi}\frac{12 - \sin^2(\mu/2)}{\sin^3(\mu/2)}$$

$$c_+^{(cell)} = \frac{\alpha L_c^2}{4\pi}\frac{12 - 9\sin^2(\mu/2)\pm 2\sin^3(\mu/2)+3/5\,\sin^4(\mu/2)}{\sin^3(\mu/2)}$$

where $\alpha$ is the bending angle per half-cell and the sign $\pm$ depends whether the central quadrupole in the cell is focusing or defocusing. $\mu_x$ and $\mu_y$ being both close to $90°$ to within $5°$, the contribution of the sum resonance turns out to be negligible compared to that of the difference resonance (1,-1): $|c_+| \sim |c_-|/N_c$, assuming $\mathbf{F}_- \sim \mathbf{F}_+$. In view of Eq. 4, the ratio between these two contributions is of the order $N_c^2\Delta^+/\Delta^- \sim 10^4$ for LHC. On the other hand non-resonant terms $d_\pm$ of Eq. 4 are shown in [1] to contribute to $Q''$ at most by 1500 units.

## 4  APPLICATION TO LHC VERSIONS 5/6

With respect to coupling, LHC versions 5 and 6 differ mostly by the tune split increased from 4 to 5 to maximise the dynamic aperture. The second-order chromaticity is found significantly reduced. The analytical model can be used to explain this observation.

LHC exhibits a super-periodicity close to 8 in $\mu_x - \mu_y$:

$$[\mu_{x_l}-\mu_{y_l}]-[\mu_{x_k}-\mu_{y_k}] = \frac{2\pi p_0}{8}\,(l-k),\; 1\le k\le l\le 8, \quad (6)$$

where $p_0 \equiv Q_x - Q_y$ is the integer tune split. Under this condition, the module of $\mathbf{F}_-$ takes the following form:

$$\mathbf{F}(p_0) \stackrel{\text{def}}{=} |\mathbf{F}_-| = \left|\sum_{k=1}^8 e^{i\frac{2\pi p_0 k}{8}}(K_2^-L)_k\right|, \quad (7)$$

It reaches a maximum when the perturbation shows an harmonic in phase with the arc phasor:

$$(K_2^-L)_k = (K_2^-L)_{max}\times\cos(2\pi p_0 k/8 + \phi),\; 1\le k\le 8, \quad (8)$$

where $(K_2^-L)_{max}$ is the tolerance on the skew sextupolar strength integrated per dipole ($3\times 10^{-3}\,\text{m}^{-2}$ for LHC) and $\phi$ an arbitrary phase, except for $p_0 = 0 \bmod 4$ where $\phi$ must be 0 or $\pi$. Using Eq.'s 7 and 8, we finally obtain

$$\begin{cases} F_{max}(p_0) = 8\,(K_2^-L)_{max} & \text{if } p_0 = 0,\,4,\,8,\,\ldots \\ F_{max}(p_0) = 4\,(K_2^-L)_{max} & \text{if } p_0 \ne 0 \bmod 4\,. \end{cases} \quad (9)$$

In terms of $Q''$, the strength of the resonance (1,-1) is then expected to be reduced by a factor 4 when going from a tune split of 4 to a tune split of 5. This is in perfect agreement with the results obtained with MAD [3]:

$$\begin{cases} Q''_{max}\sim \pm 56700\text{ (MAD)} / \pm 58500\text{ (analytic) for } p_0 = 4 \\ Q''_{max}\sim \pm 13000\text{ (MAD)} / \pm 11100\text{ (analytic) for } p_0 = 5\,. \end{cases}$$

For LHC, the tolerance on $Q''$ is 1000 units at injection [2] and 4000 units at top energy [1], the criterion being: 1) the control of the chromatic detuning and tune ripple induced, 2) the ability to accelerate/decelerate safely the *nominal* beam to measure the tune versus energy $Q(\delta)$ or the dispersion on a relevant momentum range ($\pm 2\,10^{-3}$ at injection, $\pm 5\,10^{-4}$ at collision) while avoiding an head-tail instability (that is $Q'(\delta) > 0$).

The tolerance is exceeded by about a factor of 10 both at injection and in collision, taking into account the reduced fractional tune split in collision (.01 instead of .03). Machines tend to be operated even closer to the diagonal, e.g. by a factor of 3 at HERA. There is thus a clear case for a correction of the chromatic coupling in LHC.

## 5  DECOUPLING SCHEME

### 5.1  Decoupling criteria

The correction must be as local as possible to minimise simultaneously the resonant and non-resonant terms in Eq. 4 while requiring a minimum of correctors and power supplies. Given the insignificant impact of the resonance (1,1) before correction, we choose to cancel *in each arc* the coupling coefficient $c_-$ without exciting the sum resonance.

$$\int_{\substack{k^{\text{th}} \\ \text{arc}}} ds\,\sqrt{\beta_x\beta_y}D_x\left((K_2^-)_k + (K_2^-)_k^{(cor)}\right)e^{i(\mu_x\pm\mu_y)} = 0,\; k=1..8$$

where $(K_2^-)_k^{(cor)}$ is the field distribution of the correctors in the $k^{\text{th}}$ arc. By taking advantage of the fact that the phase advances per cell are close to $90°$ in LHC, these four real conditions can be achieved with only one family of corrector per arc if the following conditions are fulfilled:

**1**. the corrector distribution must be *symmetrical* with respect to the mid-arc as it is the case for the systematic part of the error distribution. As a result, two of the four initial conditions are automatically fulfilled.

**2**. the correctors must be arranged in pairs spaced by an *odd number* of cells ($\sim m\pi + \pi/2$ in phase) in order not to

| Case | | $Q_I''$ | $Q_{II}''$ | $dQ_I/dE_I$ | $dQ_I/dE_{II}$ | $dQ_{II}/dE_{II}$ |
|---|---|---|---|---|---|---|
| LHC Version 5 | Perfect machine | 39.6 | 12.3 | 126.9 | $-1644.9$ | 451.9 |
| Tune split | worst case, no correction | $-56683.9$ | 56693.5 | $-332.4$ | $-1264.1$ | $-99.5$ |
| of 4 | correction | $-70.0$ | 107.1 | 180.9 | $-1501.6$ | 144.7 |
| LHC Version 6 | Perfect machine | 4.9 | $-1.3$ | 118.1 | $-1801.1$ | 705.2 |
| Tune split | worst case, no correction | $-13003.9$ | 12989.6 | $-103.3$ | $-1602.9$ | 436.7 |
| of 5 | correction | $-33.1$ | 55.1 | 148.2 | $-1627.6$ | 538.9 |

Table 1: Second order chromaticities and anharmonicity coefficients [m$^{-1}$] induced by the systematic component $a_3$ of the dipoles before and after correction

excite the sum resonance (1,1).

3. $\displaystyle\int_{\substack{k^{\text{th}}\\ \text{arc}}} ds \sqrt{\beta_x\beta_y}\, D_x\big((K_2^-)_k + (K_2^-)_k^{(cor)}\big)\cos(\mu_x - \mu_y) = 0$ .

4. For the chromatic correctors not to induce geometric aberrations, each corrector must be formed of a pair of skew sextupoles spaced by $2(2p+1)$ cells (i.e. $\sim$ $(2p+1)\pi$ in phase) and preferably placed close to focusing quadrupoles ($\beta_y$ small) in order to minimise the driving term of the third order resonance (0-3).
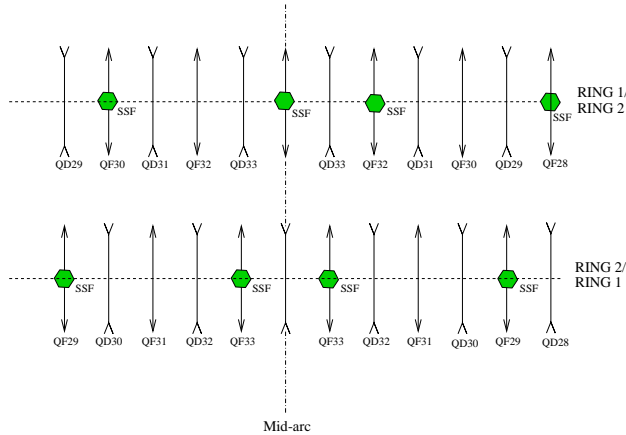
## 5.2  Optimal solution for LHC



Figure 2: Optimal scheme with 4 skew sextupolar corrector per arc (horizontal chromaticity sextupoles tilted by $90°$)

The conditions 2 and 4 impose a minimum number of 4 correctors per arc. An elegant solution is obtained by tilting by $90°$ four focusing chromaticity sextupoles (SF) carefully chosen to satisfy the four previous criteria (Fig. 2). While fully satisfying condition 4, this choice is further justified as it does not decrease the safety margin of the chromaticity correction scheme. The SF's and SD's are indeed made of the same corrector magnet. The integrated strength of the SD's sets the ultimate performance of the system as the dispersion function is smaller by a factor 2 as compared to its value at the SF's. Moreover, for this reason, the efficiency of the skew sextupolar correctors is also doubled. Finally, due to the change of polarity from arc to arc and from ring to ring, note that the condition 1 can be fully

respected in only 50% of the arcs. Nevertheless, insofar as the phase advance difference per half-cell is relatively small, this should not deteriorate the quality of the correction. This scheme was tested on the LHC Versions 5 and 6 (see Tab. 5 obtained by running the command STATIC of MAD). The quality of the correction is excellent; the amplitude detuning induced after correction is negligible.

Finally, tracking studies at injection have shown that the dynamic aperture was quite insensitive to the multipole $a_3$ and to its correction [1]. This confirms, on the one hand, that removing 32 chromaticity sextupoles from the lattice does not reduce the dynamic aperture; on the other hand, this shows that, as expected, the geometric aberrations induced by the skew sextupolar correctors SSF remain insignificant.

## 6  CONCLUSIONS

The second-order chromaticity observed numerically both on the injection and collision optics of the LHC is fully explained by the phenomenon of chromatic coupling. The size of LHC is such that this effect becomes noticeable. The main consequences are a reduction of the accessible momentum range required to measure $Q(\delta)$ and the dispersion and a tune modulation at twice the synchrotron frequency. These effects are amplified if the distance of the working point to the diagonal is reduced as is often the case in practice. Its correction requires a single family of skew sextupoles organised in two pairs per arc to allow an orthogonal control of chromatic coupling. An efficient and cost-effective solution was found by tilting some chromaticity sextupoles rather than providing additional correctors. This scheme is implemented in the latest Version 6.1 of LHC.

## REFERENCES

[1] S. Fartoukh. *Chromatic coupling induced by skew sextupolar field errors in the LHC main dipoles and its correction.* LHC Project Report 278, March 1999.

[2] J.P. Koutchouk. *Chromatic Properties of the LHC Lattice version 5.0 at Injection.* LHC Project Note 113, Oct. 1997.

[3] Hans Grote, F.Christoph Iselin (CERN). *The Mad Program Version 8.10.* CERN/SL/90–13 (AP), Jan. 1993.

[4] G. Guignard. *The conceptual theory of all sum and difference resonances in 3D magnetic field.* CERN 76-06/ISR, 1976.