VORTEX FORMATION IN RELATIVISTIC ELECTRON BEAM TRANSPORT

D. Jovanović, Institute of Physics, P. O. Box 57, Yu–11001 Belgrade, Yugoslavia
R. Fedele, Universitá di Napoli and INFN Sezione di Napoli, via Cintia, I–80126 Napoli, Italy
P. K. Shukla, Ruhr–Universität Bochum, D–44780 Bochum, Germany

Abstract

A relativistic electron beam is described in the moving frame by the electron-magnetohydrodynamic (EMHD) equations of plasma physics. For large beam currents, the accelerator magnetic field becomes unstable to the fast magnetic reconnection, and we present a plausible saturated state in the form of a complex vortex pattern.

1 INTRODUCTION

The highly intensive relativistic beam behaves predominantly as a continuous medium, rather than the collection of individual particles. The Fermilab experiments [1] revealed kinetic colective phenomena in the beam behavior such as the plasma echo, etc, known in plasma physics. The fluid description has been utilized in the study of relativistic beam phenomena, e. g. the electromagnetic filamentation [2] and the intense equilibrium flow [3]. The magnetic phenomena connected with the torsion of the magnetic flux tubes (generation of diamagnetic vortices, magnetic field reconnection, etc.) may develop in large intensity electron beams, whose radius is close to the electron collisionless skin depth d_e, where $d_e = c/\omega_{p,e}$. Such are the future $5-$TeV collider [4] (the density of its bunched beam $10^{17}\mathrm{cm}^{-3}$, corresponding to $d_e = 15.35\mu$m) and the electron beam used to energize the Dutch free-electron maser [5] ($I=$ 12A, which corresponds to $r/d_e = 1/20$). Diamagnetielectron vortices at the skin depth scale appear in inductive accelerators [6], where they are responsible for the emergence of unsteady electron flows and for the turbulent mixing of the electron flows in the beam.

We investigate the perpendicular dynamics of high intensity electron beams, whose radius is comparable with the collisionless skin depth, using the thermal equilibrium model for the particle distribution [7]. The space charge of the beam is almost fully neutralized by the effects of the self-magnetic field for the relativistic beam velocities, and the electron fluid behaves as a quasineutral plasma. It is described by the electron-magnetohydrodynamic (EMHD) equations [8, 9]. Complex magnetic geometries that are used to support the beam, containing magnetic separatrices, null- and X-points, are unstable in conductive fluids. We demonstrate that a feasible saturated state of the reconnection of the octupolar magnetic fields in an electron beam, has the form of a vortex pattern in the velocity field of the electrons. As such a pattern introduces new bifurcations in the magnetic field topology, with finer scales, this branching process is expected to continue, multiplying the number of vortices with deminishing scale size, eventually leading to the stochastization of the beam (or parts of it), in the vicinity of the original linearly unstable critical points.

2 BASIC EQUATIONS

We study the nonlinear dynamics of a relativistic, debunched electron beam in a linear accelerator, propagating along the z axis with a relativistic velocity $\vec{e}_z V$, immersed in the magnetic field $\vec{B} = \vec{B}_{\rm ext} + \vec{B}_{\rm beam}$. The velocity of an individual electron may deviate from the average beam velocity by a small, nonrelativistic, amount. The magnetic field $\vec{B}_{\rm ext}$ is produced by the currents in the coils and in the metallic vessel, and $\vec{B}_{\rm beam}$ is produced by the electric current of the electron beam. The total field is expanded as

$$\vec{B}\left(\vec{r},t\right) = \sum_n \vec{B}_n\left(r,z,t\right)\cos\left(n\theta+\varphi_n\right). \qquad (1)$$

In a linear accelerator, the dipolar ($n = 1$) component is absent. The largest amplitude is that of the quadrupole, ($n = 2$), used for the beam focusing by the periodic magnetic lenses are along the beam. We account also for small monopolar and octupolar components of the total magnetic field, which are homogeneous along the beam. The monopolar Biot-Savart's field is produced self-consistently by the beam current, while the octupolar component is partially applied externally to achieve the fine tuning of the magnetic lenses, and a part of it arises accidentally, from the small errors in the quadrupolar coils.

The evolution of the electron beam is studied in the co-moving reference frame (denoted by primes), which is described by the standard Lorentz transformations. For a highly relativistic beam with a small parallel velocity spread $|v_z - V| \ll |c - V| \ll c$, the charge density and the parallel current in the moving frame are negligibely small $\rho'/\rho \sim j_z'/j_z \sim (v_z - V)/(c - V) \ll 1$. We adopt the model of a warm fluid which is in a thermodynamic equilibrium [10], neglecting the particle diffusion due to collisions and turbulent effects. The pressure tensor is taken to be anisotropic, but purely diagonal, $\widehat{p} = nT_\perp(\vec{e}_x\vec{e}_x + \vec{e}_y\vec{e}_y) + nT_\parallel\vec{e}_z\vec{e}_z$. We also assume small, nonrelativistic, deviations of the particle velocities from the average beam velocity $\vec{e}_z V$. Then, the electron beam dynamics is described by the hydrodynamic momentum equation in the moving frame

$$\left(\frac{\partial}{\partial t'} + \vec{v}'\nabla'\right)\vec{v}' = -\frac{e}{m_e}\left(\vec{E}' + \vec{v}' \times \vec{B}' + \frac{\nabla'\hat{p}'}{n'e}\right). \tag{2}$$

The quadrupolar magnetic field of the magnetic lenses, which is periodic along the z axis $\vec{B}_2 = \vec{B}_{2,0}(r)\cos 2\theta \cos kz$, is observed in the moving frame as an electromagnetic wave, whose electric field is equal to

$$\vec{E}_2' = \vec{e}_z V \times \vec{B}_{2,0}'(r')\cos 2\theta' \cos(\omega' t' + k' z'), \tag{3}$$

where $\vec{B}_{2,0}'(r') = \vec{B}_{2,0}(r)\left(1 - V^2/c^2\right)^{-1/2}$, $k' = k\left(1 - V^2/c^2\right)^{-1/2}$, $\omega' = k'V$, $r' = r$, $\theta' = \theta$. We will conveniently separate the high- and low-frequency components of the momentum equation (2). The amplitude of the "rapid" component is much larger than that of the "slow" one, and performing the average over the rapid oscillations we readily obtain the "slow" momentum equation

$$\left(\frac{\partial}{\partial t'} + \vec{v}_s'\nabla'\right)\vec{v}_s' =$$

$$-\frac{e}{m_e}\left(\vec{E}_s' + \vec{v}_s' \times \vec{B}_s' + \frac{\nabla' p_{\perp,s}'}{n'e} - \nabla'\phi_p'\right). \tag{4}$$

The subscript s is used to denote the "slow" components and the ponderomotive potential ϕ_p' is given by

$$\phi_p' \equiv -\frac{m_e}{2e}\left\langle \vec{v}_r'^{\,2}\right\rangle = -e\,\frac{1 + \cos 4\theta'}{8 m_e k'^2}\left|\vec{B}_{2,0_\perp}'(r')\right|^2. \tag{5}$$

To avoid the solutions of the "slow" equation (4) that are secularly growing in time, the leading order curlfree term at the right-hand-side must be identically equal to zero

$$\phi_s^{(0)'} - \frac{p_{\perp,s}^{(0)'}}{en'} + \phi_p' = 0. \tag{6}$$

Eq. (6) describes the leading-order hydrodynamic stability of the electron beam. Here $\phi_s^{(0)'} = \phi_{\mathrm{beam}}^{(0)'} - V A_{s,z_{\mathrm{ext}}}^{(0)'}$ is the leading-order "slow" potential,

$$\phi_{\mathrm{beam}}^{(0)'} = \nabla_\perp'^{-2}\left[\frac{en}{\epsilon_0}\left(1 - \frac{V v_z}{c^2}\right)\left(1 - \frac{V^2}{c^2}\right)^{-1/2}\right]$$

and $A_{s,z_{\mathrm{ext}}}^{(0)'}$ is the "slow" vector-potential associated with external currents.

The condition (6) can be met by the appropriate shaping of magnetic coils. The quadrupolar magnets are designed so that the monopolar component of the ponderomotive potential produces an *inward* force that balances the beam defocusing due to the residual space charge and other effects, discussed earlier. Such an inward ponderomotive force is produced by a wiggler quadrupolar magnetic field whose amplitude has the minimum at the beam axis. Such magnetic field inevitably produces also the octupolar component of the ponderomotive potential, [see Eq. (5)], which is balanced by the fine tuning of the octupole magnets.

For the phase velocities that are much smaller than the speed of light, $|(\partial^2/\partial t^2)\vec{B}_s'| \ll c^2|\nabla_\perp'^2 \vec{B}_s'|$, we can neglect the displacement current on the slow time-scale, and use $\vec{v}_s' = -[c^2\epsilon_0/(n'e)]\,(\nabla' \times \vec{B}_s')$, while for $d/dt \ll \omega_{p,e}$ $[\omega_{p,e}^2 = n'e^2/(m_e\epsilon_0)]$ the beam density perturbation may be regarded as negligible. For the slow magnetic field which is homogeneous along the beam propagation, the curl of Eq. (4) yields the following system of two coupled scalar equations for the slow time evolution of the magnetic field

$$\left[\frac{\partial}{\partial t} + (\vec{e}_z \times \nabla_\perp B_z)\cdot\nabla_\perp\right]\left(1 - \nabla_\perp^2\right)B_z -$$

$$(\vec{e}_z \times \nabla_\perp A_z)\cdot\nabla\left(1 - \nabla_\perp^2\right)A_z = 0, \tag{7}$$

$$\left[\frac{\partial}{\partial t} + (\vec{e}_z \times \nabla_\perp B_z)\cdot\nabla_\perp\right]\left(1 - \nabla_\perp^2\right)A_z = f(t), \tag{8}$$

where $f(t)$ is an arbitrary function of time, the magnetic field is normalized to an arbitrary field B_0, $\vec{B} \to \vec{B}'/B_0$, time to the corresponding electron gyroperiod $t \to -t'eB_0/m_e$, the distance to the collisionless skin depth, $\vec{r} \to \vec{r}\,'/d_e$, and $d_e = [c^2\epsilon_0 m_e/(n'e^2)]^{1/2}$. Eqs. (7), (8) are identical to the two-dimensional electron-magnetohydrodynamic (EMHD) equations [8], which describe the fast phenomena (compared to the typical ion response time) involving the electron population in collisionless magnetized quasineutral plasmas. In our case, the role of the ions is played by the magnetic field of the beam, since for the relativistic velocities V, the Lorentz force associated with it almost fully compensates for the space charge effects, and the beam behaves as being neutralized.

For a stationary solution, the arbitrary function $f(t)$ in Eq. (8) must be set to zero. Using $\partial/\partial t = 0$, Eqs. (7), (8) take the forms of mixed products, and are readily integrated as

$$\left(1 - \nabla_\perp^2\right)A_z = \mathcal{F}(B_z), \tag{9}$$

$$\left(1 - \nabla_\perp^2\right)B_z + A_z\,\frac{d\mathcal{F}(B_z)}{dB_z} = \mathcal{G}(B_z). \tag{10}$$

Here $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions of the given argument, which in each particular case are to be determined from the appropriate boundary and continuity conditions.

3 OCTUPOLAR VORTEX

Multipolar vortices are characteristic for plasmas that in the unperturbed state feature both the velocity and magnetic shears [12]-[14]. They arise when the unperturbed magnetic field is a nonlinear functions of r, and contains higher harmonics in θ. Such fields contain also magnetic separatrices and X-points, which are known to be unstable. As a simple model, we adopt the background magnetic field in the form

$$B_z^{(0)} = D + \frac{1}{4L_z}\left(r^2 + s r^4\cos 4\theta\right), \tag{11}$$

$$A_z^{(0)} = -\frac{1}{4L_\perp}\left(r^2 + sr^4\cos 4\theta\right), \qquad (12)$$

where D is the (normalized) uniform solenoidal focusing magnetic field, L_z and $L_\perp$ are the characteristic lengths of inhomogeneities in the parallel and perpendicular direction, respectively, and the parameter s determines the amplitude of the octupolar component. The magnetic field (12) possesses an X-line in the perpendicular magnetic field at $r = 0$. Similar magnetic structures are subject to the fast magnetic reconnection instability in the plasma EMHD regime, [11].

In order to construct the octupole, we solve our basic equations (9), (10) assuming linear functions $\mathcal{F}$ and $\mathcal{G}$

$$\mathcal{F}(\xi) = F_0 + F_1\xi, \quad \mathcal{G}(\xi) = G_0 + G_1\xi, \qquad (13)$$

allowing for different values of the parameters inside and outside of the vortex core, which is a circle in the x, y plane with the radius r_0. In the external region, $r \geq r_0$, for a solution which is finite for $r \to \infty$ we have

$$F_0^{out} = \frac{1-D}{L_\perp}, \quad G_0^{out} = -\frac{1}{L_z} - D\frac{L_z^2}{L_\perp^2},$$

$$F_1^{out} = \frac{L_z}{L_\perp}, \quad G_1^{out} = 1 + \frac{L_z^2}{L_\perp^2}, \qquad (14)$$

Inside the vortex core, $r < r_0$, Eqs. (9) and (10) may be decoupled to give a fourth order, linear equation

$$\left(\nabla_\perp^2 + \kappa_1^2\right)\left(\nabla_\perp^2 + \kappa_2^2\right)\left(A_z^{in} + b\right) = 0. \qquad (15)$$

$$\kappa_1^2\kappa_2^2 = {F_1^{in}}^2 + 1 - G_1^{in}, \qquad \kappa_1^2 + \kappa_2^2 = G_1^{in} - 2, \quad (16)$$

$$b = -\frac{1}{\kappa_1^2\kappa_2^2}\left[F_0^{in}\left(1 - G_1^{in}\right) + F_1^{in}G_0^{in}\right]. \qquad (17)$$

Due to the explicit presence of the terms r^2 and $r^4\cos 4\theta$ in the unperturbed fields $A_z^{(0)}$, $B_z^{(0)}$, the perturbed fields must also involve the zeroth and fourth cylindrical harmonics

$$\delta A_z \equiv A_z - A_z^{(0)} = \delta A_{z,0} + \delta A_{z,4}\cos 4\theta,$$

$$\delta B_z \equiv B_z - B_z^{(0)} = \delta B_{z,0} + \delta B_{z,4}\cos 4\theta, \qquad (18)$$

Using Eq. (14), we can readily write the solution in terms of the Bessel functions K_0, K_4 outside, and J_0, J_4 inside the vortex core, respectively. At the edge of the vortex core, $r = r_0$, the usual continuity conditions must be satisfied for each cylindrical harmonic. We require that the functions $\mathcal{F}$ and $\mathcal{G}$ are continuous, that the core edge is an isoline of B_z with the value a, and that the functions δA_z, $(\partial/\partial r)\delta A_z$ and $(\partial/\partial r)\delta B_z$ are continuous.

A typical octupole is shown in Fig. 1. The vortex size is $\sim 10\%$ of the collisionless skin depth d_e, which is comparable with the beam radii of the devices described in [4, 5]. This kind of structure is expected to emerge as the result of the saturation of the fast magnetic reconnection in the complex geometry of the accelerator's magnetic field which possesses an X-line. The full dynamics of such a process is not studied here. It is expected to involve kinetic effects, such as the cyclotron damping of singular current layers, electron trapping, etc.

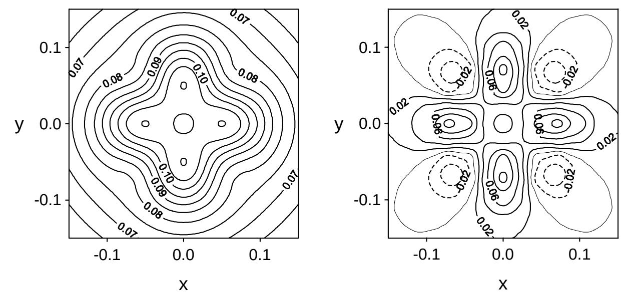

Figure 1: The perturbations of the z-component of the vector potential δA_z, and of the parallel magnetic field δB_z, associated with the octupole. The background magnetic field parameters are $L_\perp = 1$, $L_z = .9\,r_0$, $s = 2.5/r_0^2$, and the core radius is $r_0 = 0.1$.

REFERENCES

[1] L. K. Spentzouris, J. -F. Ostigui, and R. L. Colestock, Phys. Rev. Lett., **76**, 620 (1996).

[2] R. C. Davidson, B. H. Hui, and C. A. Kapetanakos, Phys. Fluids, **18**, 104 (1975).

[3] M. Reiser, Phys. Fluids, **20**, 477 (1977).

[4] B. Richter, in *Laser acceleration of particles*, (Ref. 6.), Proc. Second Workshop on Laser Acceleration of Particles, A. I. P. Conf. Proc. No. 130, edited by C. Joshi and T. C. Katsouleas, (A. I. P. New York, 1985).

[5] A. G. A. Verhoveen, W. A. Bongers. W. L. Bratman, M. Caplan, G. G. Denisov, G. van Dijk, C. A. van der Greer, P. Manintveld, A. J. Poelman, J. Pluygers, M. Yu. Shmelyov, P. H. M. Smeets, A. B. Sterk, And W. H. Urbanus, Phys. Plasmas, **5**, 2029 (1998).

[6] B. W. Church, and R. N. Sudan, Phys. Plasmas, **3**, 3809 (1996).

[7] R. C. Davidson, Phys. Plasmas, **5**, 3459 (1998).

[8] A. S. Kingsep, K. V. Chukbar, and V. V. Yan'kov, in *Reviews of Plasma Physics*, edited by B. Kadomtsev (Consultant Bureau, New York, 1990), Vol. 16, p. 243.

[9] P. K. Shukla, and L. Stenflo, Phys. Lett., **A 259**, 49 (1999).

[10] S. M. Lund, and R. C. Davidson, Phys. Plasmas, **5**, 3028 (1998).

[11] S. V. Bulanov, F. Pegoraro and A. S. Sakharov, Phys. Fluids, **B4**, 2499 (1992).

[12] D. Jovanović, F. Pegoraro and J. Juul Rasmussen, J. Plasma Phys. **60**, 383 (1998).

[13] J. Vranješ, D. Jovanović and P. K. Shukla, Phys. Plasmas, **5**, 4300 (1998).

[14] D. Jovanović, and F. Pegoraro, Phys. Plasmas, **7**, 889 (2000).