

SINGLE BUNCH BEAM BREAKUP – A GENERAL SOLUTION*

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Abstract

Caporaso, Barletta and Neil (CBN) found a solution to the problem of the single-bunch beam breakup in a linac[1]. However, their method applies only to the case of a beam traveling in a strongly betatron-focused linac under the influence of the resistive wall impedance. We suggest in this paper a method for dealing with the same problem. Our methods is more general; it applies to the same problem under any impedance, and it applies to a linac with or without external betatron focusing.

1 CBN RESULTS

We denote the location along the linac by the variable z . The beam is taken to be traveling in the positive z direction, and the entrance to the linac is located at $z = 0$. We assume throughout this paper that the charged particles are uniformly distributed longitudinally within the unperturbed bunched beam. For $z > 0$, the equation of motion for a beam particle is

$$\left(\frac{\partial^2}{\partial z^2} + k_y^2\right)y(\tau, z) = \int_0^\tau d\tau' g(\tau - \tau')y(\tau', z), \quad (1)$$

where $\tau = t - z/v$ describes the relative longitudinal position of the particle inside the bunch, v is the particle velocity, k_y is the wave number representing the betatron focusing strength, and $g(\tau)$ is the wake function. With this definition of τ , $\tau_1 > \tau_2$ implies that the particle 2 is in front of the particle 1; $\tau = 0$ corresponds to the head of the bunch. Assuming the initial condition $y_0(\tau) \equiv y(\tau, z = 0)$ and $y'_0(\tau) \equiv y'(\tau, z = 0)$ to be given, where y' denotes the derivative of y with respect to z , we wish to find $y(\tau, z)$ from the equation (1) for all $z > 0$. The wake function vanishes for $\tau < 0$, and the quantity

$$\tilde{g}(s) = \int_0^\infty d\tau e^{-s\tau} g(\tau). \quad (2)$$

is proportional to the longitudinal beam impedance.

If the source of the wakefield is the resistive wall of a circularly cylindrical beam pipe, then

$$g(\tau) = \Omega^{5/2}/(2v^2\sqrt{\tau}) \quad \text{for } \tau > 0, \quad (3)$$

$$\tilde{g}(s) = \sqrt{\pi} \Omega^{5/2}/(2v^2\sqrt{s}), \quad (4)$$

where $\Omega^{5/2} = (2eIv/\pi mc\gamma b^3)\sqrt{1/\pi\epsilon_0\sigma}$ with σ the conductivity, b the radius of the beam pipe, and γ is the relativistic energy coefficient. For the special case of a strongly focused linac with the resistive wall impedance,

* Work supported by US DOE DE-AC02-98CH10886

CBN found, by taking advantage of the specific form of the wake function (3), the solution of (1) to be

$$y(\tau, z) = \frac{d_0}{2\pi i} \int_{-i\infty}^{i\infty} \frac{ds}{s} \exp(A^2 s) \times \cos \sqrt{k_y^2 z^2 - 2k_y z/\sqrt{s}}, \quad (5)$$

where $A \equiv \sqrt{\pi\tau} z \Omega^{5/2}/4k_y v^2$, and the initial condition is taken to be $y_0(\tau) = d_0 = \text{constant}$, and $y'_0(\tau) = 0$ for all τ . Recall that to each beam particle is associated a value of τ . The asymptotic behaviour of the CBN solution (5) for the beam particle when $z \rightarrow \infty$ is[1]

$$y(\tau, z) \longrightarrow \exp[(z/l_k)^{2/3}], \quad (6)$$

with the growth length given by

$$l_k = (2/3)^{3/2} \frac{8k_y v^2}{\sqrt{\pi\tau} \Omega^{5/2}}. \quad (7)$$

The method these authors employed in finding the solution (5) does not apply to the cases of the wake function other than the specific one given by (3). Nor does the method apply to the resistive wall case when $k_y = 0$. We propose an alternative method in the next section, which is applicable to any impedance in a linac with or without betatron focusing.

2 A GENERAL SOLUTION

The case of a general wake function $g(\tau)$ is treated in this section[3]. Also, $k_y \neq 0$ is not assumed. We want to solve the transient problem (the initial value problem) of (1) corresponding to a bunch with a finite bunch length l_τ ; the beam particles satisfy $\tau \in [0, l_\tau]$.

In terms of the Laplace-transformed quantity

$$\tilde{y}(s, z) = \int_0^\infty d\tau e^{-s\tau} y(\tau, z), \quad (8)$$

the equation of motion (1) is equivalent to

$$\left(\frac{\partial^2}{\partial z^2} + k_y^2\right)\tilde{y}(s, z) = \tilde{g}(s)\tilde{y}(s, z). \quad (9)$$

Now apply another Laplace transform

$$\underline{\tilde{y}}(s, p) = \int_0^\infty dz e^{-pz} \tilde{y}(s, z). \quad (10)$$

Then y is related to $\underline{\tilde{y}}$ by an inverse Laplace transform

$$y(\tau, z) = \frac{1}{(2\pi i)^2} \int ds \int dp e^{s\tau + pz} \underline{\tilde{y}}(s, p), \quad (11)$$

where, in terms of some positive numbers s_1 and p_1 , the integration regions in s and p are $(s_1 - i\infty, s_1 + i\infty)$ and $(p_1 - i\infty, p_1 + i\infty)$, respectively; the Bromwich contours are understood. Combining (9) and (10), we have

$$\tilde{y}(s, p) = \frac{s\tilde{y}_0(s) + \tilde{y}'_0(s)}{p^2 + k_y^2 - \tilde{g}(s)}, \quad (12)$$

where $\tilde{y}_0(s)$ and $\tilde{y}'_0(s)$ are, respectively, the Laplace transforms of $y_0(t)$ and $y'_0(t)$. Substituting the last equation into (11) and performing the integration in the p variable, we obtain the following general transient solution to the single-bunch beam breakup problem:

$$y(\tau, z) = \frac{1}{2\pi i} \int_{s_1 - i\infty}^{s_1 + i\infty} ds e^{s\tau} [\tilde{y}_0(s) \cos k_c z + \frac{1}{k_c} \tilde{y}'_0(s) \sin k_c z], \quad (13)$$

where the coherent wave number k_c is a function of s given by

$$k_c^2(s) = k_y^2 - \tilde{g}(s). \quad (14)$$

Note that a term which is exponentially smaller in the asymptotic limit $z \rightarrow \infty$ has been ignored inside the square bracket of (13).

Suppose the initial condition is $y'_0(\tau) = 0$ for all τ , and

$$y_0(\tau) = \begin{cases} d_0, & \text{if } l_\tau > \tau > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (15)$$

then this condition is for our purpose equivalent to the condition

$$y_0(\tau) = \begin{cases} d_0, & \text{if } \infty > \tau > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (16)$$

since from causality, the fictitious particles of the last condition in the range $\infty > \tau > l_\tau$ can not influence the motion of the beam particles in the range $0 < \tau < l_\tau$ (tail can not affect the head.) Thus we can set

$$\tilde{y}_0(s) = d_0 \int_0^\infty d\tau e^{-s\tau} = d_0/s, \quad (17)$$

and the transient solution (13) simplifies to

$$y(\tau, z) = \frac{d_0}{2\pi i} \int_{s_1 - i\infty}^{s_1 + i\infty} \frac{ds}{s} e^{s\tau} \cos k_c(s)z. \quad (18)$$

This is the general solution to the single-bunch, beam-breakup problem corresponding to the initial condition $y'_0(\tau) = 0$ for all τ , and (15).

3 RESISTIVE WALL CASE

In this section we discuss the solution (18) when the wake-field is that due to the resistive beam-chamber wall for both the cases with and without external focusing represented by

k_y . Let us start by repeating the solution that was derived above,

$$y(\tau, z) = \frac{d_0}{2\pi i} \int_{s_1 - i\infty}^{s_1 + i\infty} \frac{ds}{s} e^{s\tau} \cos k_c(s)z, \quad (19)$$

and

$$k_c^2(s) = k_y^2 - \sqrt{\pi} \Omega^{5/2} / (2v^2 \sqrt{s}). \quad (20)$$

It is instructive to see how the above solution is related to the CBN solution (5). If we perform a change of variables

$$s \longrightarrow \left[\frac{\sqrt{\pi} z \Omega^{5/2}}{4k_y v^2} \right]^2 s \quad (21)$$

in the equations (19) and (20), we obtain the equation (5). Note that the last transformation of variable is singular at $k_y = 0$. This explains why, while the solution (19) with (20) is applicable to the case of vanishing k_y , the CBN solution is not.

We calculate now the growth length of the resistive-wall case when there is no external betatron focusing: $k_y = 0$. The asymptotic behavior for large z of our solution (19) with

$$k_c^2(s) = -\sqrt{\pi} \Omega^{5/2} / (2v^2 \sqrt{s}) \quad (22)$$

can be obtained by the method of steepest descent. The result is

$$y \longrightarrow \exp[(z/l_0)^{4/5}], \quad (23)$$

where the growth length is,

$$l_0 = \frac{4}{5^{5/4} a^{1/2} \tau^{1/4}}, \quad (24)$$

and

$$a \equiv \frac{\sqrt{\pi} \Omega^{5/2}}{2v^2}. \quad (25)$$

We give in the next section a numerical example to illustrate the results of this section.

4 EXAMPLE

Consider the example of an electron beam passing through a cylindrical beam pipe in a wiggler magnet of peak magnetic field $B_W = 0.65$ Tesla. The conductivity and the radius of the beam pipe are, respectively, $\sigma = 10^6$ /(Ohm m) and $b = 2.5$ mm. The beam energy is 250 MeV, the beam current $I = 300$ A, and the bunch length $l_\tau = 10$ ps.

Let us first consider the case of a planar wiggler. The magnetic field is in the y direction and $B_x = 0$. The beam is betatron focused in the y direction but not in the x direction. Substituting these numbers into the formulae of the previous sections, we obtain

$$l_y = 46 \text{ m} \quad (26)$$

and

$$l_x = 3 \text{ m}. \quad (27)$$

The result $l_y \sim 15 l_x$ for this example indicates that for a planar wiggler, the contrast of the magnitudes of the beam instabilities in the two directions can be quite striking.

Consider now an alternative wiggler. It has been pointed out[4] that by adopting a wiggler with a suitably chosen parabolic surface, the focusing effects in the x and the y directions can be equalized. If the same magnetic field $B_W = 0.65$ Tesla is chosen for such a wiggler, we obtain $l_x = l_y = 46/\sqrt{2} = 33$ m.

5 CONCLUSION

We found a general solution to the problem of single-bunch beam breakup in a periodic linac. The results for the case of the resistive-wall wakefield is presented in detail, and the reason for the non applicability of the CBN method for the case of zero-focusing resistive-wall case is given. The single-bunch beam-breakup problem caused by other forms of the impedance will be presented elsewhere[5].

REFERENCES

- [1] G.J. Caporaso, W.A. Barletta and V.K. Neil, Particle Accelerators, **11**, 71(1980). For a higher order correction to the results of this paper, see the next reference.
- [2] D.H. Whittum, A.M. Sessler and V.K. Neil, Phys.Rev. A **43**, 294(1991)
- [3] The method of this section was used previously in G. Decker and J.M. Wang, Phys. Rev. D, **38**, 980(1988) for a cumulative bunched beam breakup problem.
- [4] E.T. Scharlemann, J. Appl. Phys. **58**, 2154(1985)
- [5] S.R. Mane, N. Towne and J.M. Wang, in preparation.