PARTICLE DIFFUSION DUE TO COULOMB SCATTERING

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Abstract

Conventionally, the multiple and single particle scattering in a storage ring are considered to be independent. Such an approach is simple and often yields sufficiently accurate results. Nevertheless, there is a class of problems where such an approach is not adequate and the single and multiple scattering need to be considered together. This can be achieved by solving an integro-differential equation for the particle distribution function, which correctly treats particle Coulomb scattering in the presence of betatron motion. A derivation of the equation is presented in the article. A numerical solution for one practical case is also considered.

1 DIFFUSION EQUATION

Let $f(x,\theta,t)$ be the one-dimensional beam transverse phase-space distribution function at time t. In the presence of damping and diffusion the evolution of the function f in a ring can be described by a Fokker-Planck equation:

$$\frac{\partial f}{\partial t}+\mathrm{v}_0\theta\frac{\partial f}{\partial x}+\kappa(t)\frac{\partial f}{\partial \theta}=\lambda(t)\frac{\partial}{\partial\theta}\theta f+\frac{D_\theta(t)}{2}\frac{\partial^2 f}{\partial\theta^2}\quad . \quad (1)$$

Here the periodic functions $\kappa(t)$, $\lambda(t)$ and $D_\theta(t)$ describe the focusing, damping and diffusion in the ring, and $\mathrm{v}_0=\beta_0 c$ is the average beam velocity. Making a transition to the action-phase variables (I, ϕ) and performing averaging over the ring circumference one obtains,

$$\frac{\partial f}{\partial t}-\lambda\frac{\partial}{\partial I}(If)=D\frac{\partial}{\partial I}\left[I\frac{\partial f}{\partial I}\right]\quad . \quad (2)$$

We define the action to be equal to the single particle emittance,

$$I=\frac{1+\alpha(s)^2}{\beta(s)}x^2+2\alpha(s)x\theta+\beta(s)\theta^2\quad , \quad (3)$$

where $\beta(s)$ and $\alpha(s)$ are the beta- and alpha functions of a ring, and s is the longitudinal beam coordinate.

For the residual gas scattering the diffusion coefficient is well known [1]. In the case of scattering in a ring it can be expressed in the following form,

$$D=4\pi r_e^2\mathrm{v}_0\left(\frac{m_e c^2}{p\mathrm{v}_0}\right)^2\sum_i Z_i(Z_i+1)L_c^i\oint n_i\beta(s)\frac{ds}{L}\,, \quad (4)$$

where the summing is performed over partial densities of residual gas atoms, and the integration averages the gas density weighted by the β-function. For the high energy scattering ($\beta_0>\alpha Z_i$, $\alpha\approx1/137$) the Coulomb logarithm is

$$L_c^i=\ln\left(\frac{\theta_{\max}^i}{\theta_{\min}^i}\right),\ \theta_{\min}^i\approx\alpha Z_i^{1/3}\frac{m_e c}{p}\,,\theta_{\max}^i\approx\frac{274}{A_i^{1/3}}\frac{m_e c}{p}, \quad (5)$$

where the minimum and maximum angles are determined by the field screening due to atomic electrons and by the diffraction on nuclei.

The solutions of Eq. (2) are commonly used to describe an emittance growth in particle accelerators due to various random diffusion processes. This equation describes well the core of the beam distribution, but completely fails to describe its tail [2]. Far away tails can be sufficiently well estimated using a single scattering approximation, but in many applications a prediction of tails behavior in vicinity of the core is required. It is possible to computer-model the distribution function by Monte-Carlo methods. However, we found it beneficial to advance the analytical treatment of the Coulomb scattering process to a point, where, for a given residual gas pressure, the corrected distribution function can be obtained with the help of a simple computer code. Similar approach was used in Ref. [3] to analyze the longitudinal diffusion due to intrabeam scattering (IBS) in a ring with laser cooling. In what follows we consider the Coulomb scattering on the residual gas, but the developed theory can be easily adapted to other Coulomb scattering phenomena.

2 COLLISION INTEGRAL

To simplify formulas we omit the summation over different gas species below. Neglecting atomic electrons and nuclear form-factor one can write the differential small angle cross-section in the following form:

$$\frac{d\sigma}{d\Omega}\approx4Z^2r_e^2\left(\frac{m_e c^2}{\mathrm{v}_0 p}\right)^2\frac{1}{\left(\theta_\perp^2+\theta_{\min}^2\right)^2}\,, \quad (6)$$

where $\theta_\perp^2=\theta_x^2+\theta_y^2$. After integrating this over θ_y and then over θ_x one obtains the one-dimensional cross-section,

$$\frac{d\sigma}{d\theta_x}\approx2\pi Z^2r_e^2\left(\frac{m_e c^2}{\mathrm{v}_0 p}\right)^2\frac{1}{\left(\theta_x^2+\theta_{\min}^2\right)^{\frac{3}{2}}}\quad , \quad (7)$$

and the total cross-section

$$\sigma_{tot}\approx\frac{4\pi Z^2r_e^2}{\theta_{\min}^2}\left(\frac{m_e c^2}{\mathrm{v}_0 p}\right)^2\quad . \quad (8)$$

For a combined treatment of both the small- and large-angle scattering one has to write the right-hand side of Eq. (2) in a general form of the collision integral [4]:

$$\frac{\partial f}{\partial t}-\lambda\frac{\partial(If)}{\partial I}=\left\langle\int_{-\infty}^{\infty}\left(\left.\frac{d\sigma}{d\theta_x}\right|_{(\theta-\theta')}-\sigma_{tot}\delta(\theta-\theta')\right)n\mathrm{v}_o f d\theta'\right\rangle_{\phi,s} \quad (9)$$

where $\delta(\ldots)$ is the Dirac's delta-function. Expressing particle angles and coordinates through the action-phase variables, neglecting $\theta_{\min}$ in the cross-section, and denoting

$$B=2\pi r_e^2\mathrm{v}_0\left(\frac{m_e c^2}{p\mathrm{v}_0}\right)^2Z^2\oint n\beta ds/L\quad . \quad (10)$$

one obtains for the collision integral,

$$\left\langle n\mathrm{v}_0 \int_{-\infty}^{\infty} \frac{d\sigma}{d\theta_x}\bigg|_{(\theta-\theta')} f \mathrm{d}\theta' \right\rangle\bigg|_{\phi s} = \tag{11}$$

$$\frac{B}{2\pi}\int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \int_0^{\infty} dI' f(I') \frac{\delta\left(\sqrt{I}\cos\phi - \sqrt{I'}\cos\phi'\right)}{\left|\sqrt{I}\sin\phi - \sqrt{I'}\sin\phi'\right|^3} \ .$$

A lengthy integration yields,

$$\left\langle n\mathrm{v}_0 \int_{-\infty}^{\infty} \frac{d\sigma}{d\theta_x}\bigg|_{(\theta-\theta')} f \mathrm{d}\theta' \right\rangle\bigg|_{\phi, s} = B\int_0^{\infty} \frac{I+I'}{|I-I'|^3} f(I') dI' \ . \tag{12}$$

Neglecting $\theta_{\min}$ in transition from Eq.(9) to Eq.(11) causes divergence of the integral at $I \approx I'$, when the scattering angle is small. Instead of performing an exact integration one can fix this divergence by modifying the kernel of Eq.(12) similar to the method used to limit divergence in Eq.(6). Combining Eqs. (9) and (12) one obtains finally the equation describing the particle scattering:

$$\frac{\partial f}{\partial t} - \lambda \frac{\partial}{\partial I}(If) = \int_0^{\infty} W(I, I') f(I', t) \mathrm{d}I' , \tag{13}$$

where the kernel W is:

$$W(I,I') = B\frac{I + I' + \dfrac{I_{\min}}{2}}{\left((I-I')^2 + (I+I')I_{\min} + \dfrac{I_{\min}^2}{4}\right)^{3/2}} - \tag{14}$$

$$- \sigma_{tot}\bar{n}\mathrm{v}_o \delta(I - I') ,$$

and $\bar{n}$ is the average gas density in the ring. The particle conservation requires that

$$\int_0^{\infty} W(I, I') dI' = 0 \quad . \tag{15}$$

That yields a relationship between the minimum action and the total interaction cross-section

$$I_{\min} = \frac{2B}{\sigma_{tot}\bar{n}\mathrm{v}_o} \cdot \tag{16}$$

Similarly to Eq. (2), Eq. (13) has a logarithmic accuracy, but unlike Eq. (2) it correctly describes the large angle scattering. Note that the form of the kernel, $W(I, I')$, presumes that the maximum angles in the beam are smaller than the maximum scattering angle $\theta_{\max}$, which is well justified in most practical cases. Otherwise $\theta_{\max}$ has to be explicitly taken into account in Eq. (6).

The accelerator aperture is always finite. Therefore the upper limit in the integral of Eq. (13) should be replaced with the boundary action value, I_b. That also yields that the distribution function at the boundary is zero, $f(I_b,t)=0$.

It is now quite trivial to obtain a Fokker-Planck equation from Eq. (13) by expanding the function f in series at $I'{\approx}I$, $f(I',t) \approx f(I,t) + f'(I,t)(I'-I) + 1/2\ f''(I,t)(I'-I)^2$, and integrating to $I_{\max} = \bar{\beta}\theta_{\max}^{\ 2}$, where $\bar{\beta}$ is the average ring β-function. The integration yields:

$$\frac{\partial f}{\partial t} - \lambda \frac{\partial}{\partial I}(If) \approx B\ln\left(\frac{I_{\max}}{I_{\min}}\right)\frac{\partial}{\partial I}\left[I\frac{\partial f}{\partial I}\right] \ . \tag{17}$$

Recalling that $\ln(I_{\max}/I_{\min})=2L_c$ we arrive at Eq. (2).

3 NUMERICAL METHOD

For a numerical solution we use a finite difference algorithm. We split the total range of the action variable, $[0,I_b]$, into N equal size cells, $\delta I = I_b/N$. Then, Eq. (13) can be rewritten as

$$\delta f_n = \frac{\delta t}{\delta I}\left(\lambda\frac{f_{n+1}I_{n+1} - f_{n-1}I_{n-1}}{2} + B\sum_{m=0}^{N-1}\tilde{W}(n,m)f_m\right) \tag{18}$$

Taking into account that the cell size is much larger than the minimum action, $I_{\min}$, we can write the probability of a particle exchange for two distant cells

$$\tilde{W}(n,m) = \frac{n+m}{|n-m|^3} , \quad n \neq m, m \pm 1 \ . \tag{19}$$

Finding the probability of the particle exchange for nearby cells is complicated by the fact that the kernel of Eq. (13) experiences strong variations on one cell size. In this case the probability of the particle exchange between two neighboring cells is equal to,

$$\overline{\delta I \frac{df}{dt}}\bigg|_{n+1\to n} \approx \int_{I_n-\delta I/2}^{I_n+\delta I/2} dI \int_{I_n+\delta I/2}^{I_n+3\delta I/2} dI' \{W(I,I')\left(f_n + (I'-I_n)f'\right) - \tag{20}$$

$$W(I',I)\left(f_n + (I-I_n)f'\right)\} = f' \int_{I_n-\delta I/2}^{I_n+\delta I/2} dI \int_{I_n+\delta I/2}^{I_n+3\delta I/2} W(I,I')(I'-I) dI' .$$

where we presumed that the particle density changes linearly across the cells, and $I_n=n\delta I$. Performing integration and taking into account that $\delta I >> \sqrt{I_n I_{\min}}$ we obtain

$$\overline{\delta I \frac{df}{dt}}\bigg|_{n+1\to n} = 2f' I_n L_I \ , \quad L_I = \ln\left(\frac{\delta I}{\sqrt{2I_n I_{\min}}}\right) \ . \tag{21}$$

This yields that the probability of particle exchange for nearby cells is

$$\tilde{W}(n, n\pm 1) = L_I(2n \pm 1) \ . \tag{22}$$

The probability $\tilde{W}(n,n)$ is determined by the particle conservation so that,

$$\sum_{m=0}^{\infty}\tilde{W}(n,m) = 0 \ \cdot \tag{23}$$

The index m in this sum is running to infinity. It takes into account that a particle can be scattered outside of the accelerator aperture. Consequently, the particle number is not conserved in a finite aperture of a ring. Note that the probability of Eq.(20) expresses the difference analog of Eq. (2) with diffusion coefficient equal to DL_I / L_c.

4 EXPERIMENTAL RESULTS

The above method has been applied to the particle scattering in Tevatron. First, using the beam scraping it was verified that the particle distribution of a 150 GeV proton beam, injected into Tevatron, is very close to a Gaussian one. Second, a new beam was injected. It was unbunched to exclude the beam heating by the RF noise and the intrabeam scattering. The beam intensity was sufficiently small to make sure that the coherent effects did not affect the beam dynamics. Third, the beam was scraped horizontally and vertically. The scraping time of a few minutes is much shorter than a characteristic time of the beam evolution. That and knowing the number of particles removed by the vertical scraping (~25%), allowed us to know the initial vertical particle distribution function. Fourth, the beam scrapers were removed and the beam was left untouched for one hour. Then, we moved the vertical scraper in, while recording the beam intensity as a function of the scraper position. The vertical scraper was chosen so that the beam momentum spread would not affect the measurements.

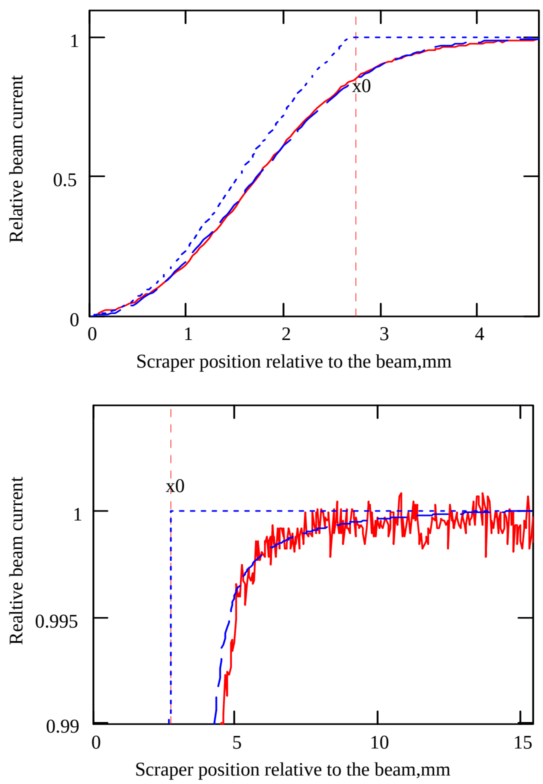

Figure 1: Dependence of the beam current on the vertical scraper position for the beam core (top) and beam tails (bottom); solid line – measurements, dashed line – computer simulations for L_I = 8.6, dotted line – the dependence which would be measured with the initial distribution; x0 – the final scraper position at the initial scraping.

The results of the measurements and the comparison with numeric simulations are shown in Figure 1. Taking into account that only one fitting parameter, the unknown average Tevatron vacuum, is used there is a good agreement between the theory and the measurements. Note that although the Coulomb logarithm is not a well-determined value and depends on Z its uncertainty does not exceed 10-20%. The experimentally determined value, L_i, coincides with the theoretical one within 5% for Z=7. The measured 5 mm×mrad/hour emittance growth rate (95%, normalized) corresponds to an average Tevatron vacuum of $3{\cdot}10^{-9}$ Torr (N_2 equivalent). Note that high accuracy of the beam current measurements allowed us to measure tiny tails of the distribution function, which could not be seen by regular beam profile monitors. If the large angle scattering is switched off in the simulations, so that particle scattering is described by diffusion only, there is large difference between calculations and measurements as presented in Figure 2.

A good agreement between the observed and the predicted distribution function tails yields an important practical conclusion that, presently, the gas scattering is the major beam heating mechanism in the Tevatron. If other than the Coulomb scattering heating mechanisms were present, the tails population would be smaller than predicted. Therefore the planned improvement of Tevatron vacuum should significantly improve the beam emittance lifetime (currently about 30 hours).

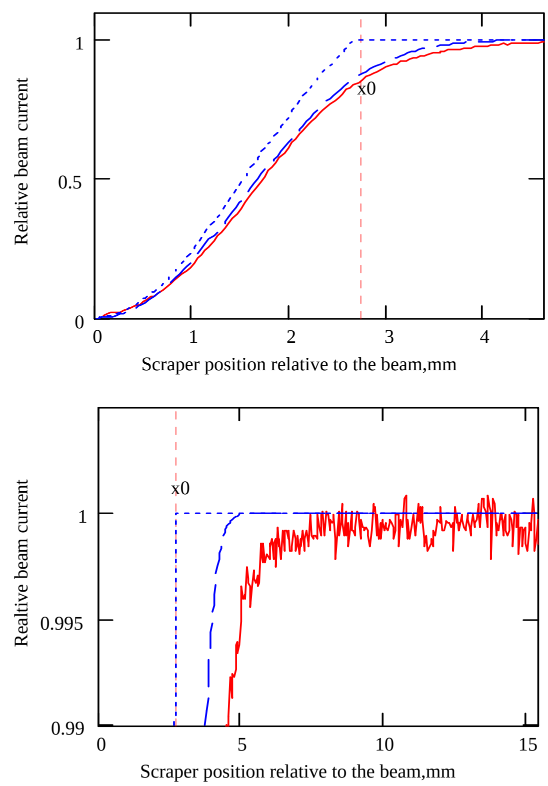

Figure 2: The same as Figure 1 but the large angle scattering is switched off in the model

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6 REFERENCES

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