PARAMETRIC RESONANCES IN INTENSE HADRON BEAMS

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Abstract

This paper considers the resonant beam instability induced by the periodic nature of external focusing force. Firstly, an eigenvalue equation that determines the frequencies of collective oscillation modes in a one-dimensional beam is given by solving the Vlasov-Poisson equations. Approximate formulae for the resonance stopbands and growth rates are derived. It is shown that the beam may become unstable when a coherent tune is close to either an integer or a half integer. Secondly, a general Hamiltonian formalism is constructed for the study of two-dimensional space-charge-dominated beams in circular accelerators. The theory suggests the possibility of a novel collective resonance driven by momentum dispersion and space charge. The particle-in-cell simulation technique is employed to confirm the existence of a "dispersive resonance" stopband.

1 RESONANT INSTABILITY IN A ONE-DIMENSIONAL BEAM

In this section, we discuss the resonant instability of a one-dimensional (1D) beam propagating through an arbitrary periodic lattice [1].

1.1 Eigenvalue Equation

In order to evaluate the coherent tunes of a 1D beam, we here assume the truncated waterbag distribution defined by

$$f_0(J) = \frac{N}{2\pi\lambda}\left[1+\mathrm{sgn}\left(\frac{\lambda}{2}-J\right)\right], \qquad (1)$$

where J is the Courant-Snyder invariant including the linear space-charge detuning, N is the number of particles per unit length, and λ is a constant corresponding to the emittance of the beam. The perturbing distribution function δf obeys the linearized Vlasov equation

$$\frac{\partial \delta f}{\partial s} + p_x\frac{\partial \delta f}{\partial x} - \frac{\partial H_0}{\partial x}\frac{\partial \delta f}{\partial p_x} = \frac{2\pi\varepsilon_0 K_{sc}}{qN}\frac{\partial f_0}{\partial p_x}\frac{\partial \delta\phi}{\partial x}, \qquad (2)$$

where q is the charge state of particles, K_{sc} is the beam perveance, H_0 is the unperturbed Hamiltonian of the beam motion, and the perturbing space-charge potential $\delta\phi$ satisfies the Poisson equation

$$\frac{\partial \delta E}{\partial x} = \frac{q}{\varepsilon_0}\int \delta f dp_x. \qquad (3)$$

Applying the linear approximation to H_0, we can solve these coupled equations analytically; Equations (1)-(3) are reduced to an eigenvalue problem [1]:

$$\sum_{n=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} M_{mn}^{k\ell}\hat{g}_n^{(\ell)} = \nu\hat{g}_m^{(k)}. \qquad (4)$$

The matrix elements are given by $M_{mn}^{k\ell} = (k+m\nu_x)\delta_{k\ell}\delta_{mn} + m\nu_x B_{k-\ell}F_{mn}$ where ν_x is the space-charge-depressed *incoherent* tune, $F_{mn} = -32/[(m-n)^2-1][(m+n)^2-1]$ for $m+n=$ even (otherwise, $F_{mn}=0$), and all information about the lattice design have been contained in the parameter B_k. Denoting β_x to be space-charge-modified betatron function, we can define B_k as

$$B_k = \frac{K_{sc}}{4\pi^2\lambda^{1/2}}\int_0^{2\pi}\beta_x^{3/2}e^{-ik\theta}d\theta \quad \text{with} \quad \theta = \int\frac{ds}{\nu_x\beta_x}. \qquad (5)$$

1.2 Resonance stopbands

Provided that the external focusing force is uniform along the beam orbit, all the eigenvalues of Eq. (4) are always real, which means that the beam is stable regardless of its density. By contrast, the tunes of several coherent modes can be imaginary in a strong focusing channel where β_x varies periodically. When the tune of one mode m_1 approaches that of another mode m_2, the system can be unstable. According to Eq. (4), such a situation takes place in the range

$$\nu_c - \delta\nu_x < \nu_x < \nu_c + \delta\nu_x, \qquad (6)$$

where

$$\nu_c = -\frac{1}{m_1-m_2}\left[k_1-k_2+\nu_x B_0(m_1F_{m_1}-m_2F_{m_2})\right], \qquad (7)$$

$$\delta\nu_x = \frac{2\nu_x\sqrt{|m_1m_2|}}{|m_1-m_2|}\left|B_{k_1-k_2}\right|\left|F_{m_1m_2}\right|. \qquad (8)$$

The approximate eigenvalues of a mth-order stable mode can be calculated from the diagonal element of the matrix $M_{mn}^{k\ell}$:

$$\nu = m\nu_x(1+B_0F_m) + \text{integer}. \qquad (9)$$

Figure 1 shows an example tune diagram where a simple FODO cell of 1m long with the filling factor of 0.5 has been assumed. The phase advance at zero current has been adjusted to be 108 degrees.

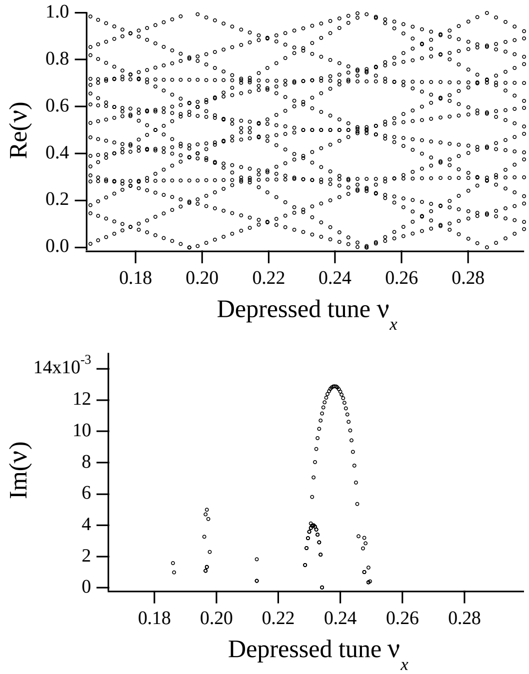

Figure 1: Frequencies of coherent oscillation modes calculated from the eigenvalue equation (4).

In many cases, severe instabilities are caused by the coupling of m and $-m$ modes. For instance, the largest stopband located in the region around $\nu_x \approx 0.24$ in Fig. 1 is due to the interaction of $m = \pm 2$ modes. Putting $m_1 = -m_2 \equiv m\,(>0)$, Eq. (7) is simplified to $\nu_c = (k_2 - k_1)/2m - \nu_x B_0 F_m$. The location of the mth-order stopband is thus determined by the relation

$$m\nu_x (1 + B_0 F_m) = \frac{k_2 - k_1}{2}, \tag{10}$$

where the left hand side is identical to Eq. (9) except for the integer offset. Note that there is a factor 2 in the denominator of the right hand side. In past theories [2], a resonance condition is derived by simply equating a coherent tune to integer. Equation (10), however, says that the mth-order resonance occurs not only when a coherent tune is close to an integer but also when it is near a half integer.

2 DISPERSIVE RESONANCE IN A TWO-DIMENSIONAL CIRCULATING BEAM

Parametric resonances in coasting hadron beams are studies in this section. In particular, we focus on a novel resonance mechanism associated with momentum dispersion and space charge [3].

2.1 Hamiltonian Formalism

The transverse spatial coordinates (x, y) of a single particle in a circular machine can be separated into the closed orbit distortion and betatron oscillation about it; namely, $x = \tilde{x} + \sum_{k=1}^{\infty} D_x^{(k)} W^k$ and $y = \tilde{y} + \sum_{k=1}^{\infty} D_y^{(k)} W^k$ where $(\tilde{x}, \tilde{y})$ are the betatron coordinates, $D_x^{(k)}$ and $D_y^{(k)}$ are horizontal and vertical dispersion functions of kth order, and W represents the energy deviation from its design value. Taking only quadrupole and horizontal bending magnets into account, the betatron Hamiltonian for a stationary coasting beam can be written as [3]

$$\tilde{H} = \frac{\tilde{p}_x^2 + \tilde{p}_y^2}{2} + \frac{1}{2}(K_x \tilde{x}^2 + K_y \tilde{y}^2) + \frac{2\pi\varepsilon_0 K_{sc}}{Nq}\left[\tilde{\phi} - \left(\frac{\partial\tilde{\phi}}{\partial\tilde{x}}\right)_{\substack{\tilde{x}=0\\ \tilde{y}=0}} \tilde{x}\right], \tag{11}$$

where K_x and K_y are the horizontal and vertical focusing functions, and the space-charge potential $\phi(x, y)$ has been expressed in the new coordinate variables: $\tilde{\phi} = \phi\left(x = \tilde{x} + \sum_{k=\mathrm{odd}} D_x^{(k)} W^k,\ y = \tilde{y};\ s\right)$. Assuming a Gaussian charge density of elliptical symmetry, Eq. (11) can be expanded as

$$\begin{aligned}\tilde{H} = {} & \frac{\tilde{p}_x^2 + \tilde{p}_y^2}{2} + \frac{1}{2}(K_x - K_{sc}\xi_{20})\tilde{x}^2 + \frac{1}{2}(K_y - K_{sc}\xi_{02})\tilde{y}^2 \\ & - \frac{K_{sc}\xi_{40}}{24}\left[\tilde{x}^4 + 4D_x^{(1)} W \tilde{x}^3 + 6(D_x^{(1)})^2 W^2 \tilde{x}^2\right] - \frac{K_{sc}\xi_{04}}{24}\tilde{y}^4 \\ & - \frac{K_{sc}\xi_{22}}{4}\left[\tilde{x}^2 + 2D_x^{(1)} W \tilde{x} + (D_x^{(1)})^2 W^2\right]\tilde{y}^2 + \cdots,\end{aligned} \tag{12}$$

where ξ_{mn}'s are periodic functions depending on the beam size.

Equation (12) indicates the possibility that nonlinear resonances may be induced in circular machines by the existence of momentum dispersion. It is also recognized, from Eq. (12), that the nonlinear dispersion is unimportant.

2.2 Simulation Results

In order to check out whether dispersive resonance stopbands really exist, we here employ the tracking code “SIMPSONS” [4]. This code enables one to simulate two- or three-dimensional motion of charged particle beams circulating in synchrotrons and storage rings. The PIC algorithm is used to analyze Coulomb interactions among macro particles. Space-charge forces are treated as discrete kicks distributed all around the machine. The scalar potential is calculated at each time step out of 10000 macro particles whose distribution evolves in a self-consistent manner as time goes on.

Among a wide range of choices we take KEK Proton Synchrotron (KEK PS) as a test lattice [5]. It has a 4-fold symmetry with 28 FODO cells in total. Each superperiod is composed of 7 FODO cells, 5 of which contain two bending magnets and the other 2 has one. The horizontal and vertical betatron functions are quite regular all over the machine. The design tunes are chosen slightly above 7 in both transverse directions. The betatron phase advance per single FODO cell is thus a little more than 90 degrees. Because of the superperiodicity of 4, there are no intrinsic structure resonances (up to fourth order) near the operating tunes except for the coupling resonances of space-charge origin. Equation (12) indicates that there is an intrinsic nonlinear coupling term proportional to $\tilde{x}\tilde{y}^2$. Following conventional understandings, such an odd-order transverse coupling is not supposed to exist unless the lattice contains a sextupole magnet or the beam has an asymmetric distribution. Since neither sextupole magnets nor field imperfections are introduced in our test lattice, we can utilize the resonance line $\nu_x - 2\nu_y = \text{integer}$ to ascertain whether the effect of the dispersive term is visible; we gradually increase the field gradient of quadrupole magnets, letting the ring operating point cross the line $\nu_x - 2\nu_y = -8$. The starting operating point (7.10, 7.75) is moved toward the final point (7.50, 7.70) in 4000 turns. In what follows, the normalized root-mean-squared (rms) emittances are initially set at 1.96πmm.mrad in the horizontal plane and 0.51πmm.mrad in the vertical plane.

First of all, we performed test simulation runs, assuming 500 MeV proton beams with no momentum spread. In this case, no instability occurred at least up to the nominal intensity of 2×10^{12} ppp. Another series of simulations was also executed under the condition of zero intensity but finite momentum spreads, and we again observed no emittance growth. These results are regarded as an evidence that there is no intrinsic *x-y* coupling along the operation line considered here as long as the space-charge potential is absent.

We now supply finite values to the beam intensity and momentum spread simultaneously. A typical emittance evolution is demonstrated in Fig. 2 where we have used the intensity $n_p = 2\times 10^{12}$ ppp and 1σ momentum spread $\sigma_{\delta p/p} = 0.002$. A considerable emittance exchange takes place only within a limited period when the operating point is passing through the stopband of $\nu_x - 2\nu_y = -8$. In Fig. 3, we show how the final levels of the horizontal and vertical emittances change with initial momentum spread. The emittance transfer has become maximum at around $\sigma_{\delta p/p} = 0.002$ in the present parameter setup. Note that the relation $2\Delta\varepsilon_x + \Delta\varepsilon_y \approx 0$, where $\Delta\varepsilon_x$ and $\Delta\varepsilon_y$ are the amounts of the emittance variations in the horizontal and vertical directions, has been approximately satisfied. This strongly suggests that the instability was caused by the coupling resonance of $\nu_x - 2\nu_y = \text{integer}$.

Considering the fact that such an effect was invisible either with no momentum spread or at zero current, the driving term responsible for this resonance is essentially related to both space-charge potential and dispersion.

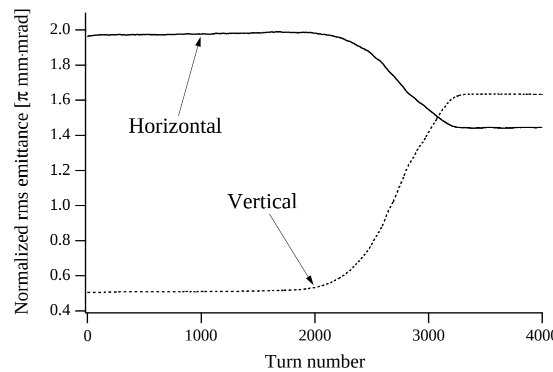

Figure 2: Time evolution of horizontal and vertical rms emittances. We have assumed a 500 MeV proton beams with $n_p = 2\times 10^{12}$ ppp and $\sigma_{\delta p/p} = 0.002$.

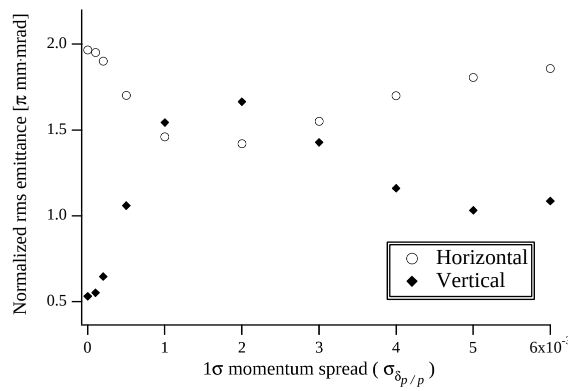

Figure 3: Final rms emittance vs. momentum spread. The values of horizontal and vertical emittances after the operating point of the ring reached (7.50, 7.70) have been plotted as a function of mometum spread. The nominal beam intensity $n_p = 2\times 10^{12}$ ppp has been assumed.

3 REFERENCES

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