BETATRON-FUNCTION MEASUREMENT IN LATTICES WITH 90° SECTIONS*.

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Abstract

The paper describes the extension of the previously published lattice-function measurement for lattices with 90° phase-advance/cell sections to include error treatment and propagation and an alternate method.

1 INTRODUCTION

Lattice diagnostics for the PEP-II Low Energy Ring (LER) has been described in a previous paper.[1] The LER has 90° phase advance in the arcs, making straightforward application of the formulae used at CERN-LEP[2] impossible. We therefore developed an algorithm that will calculate meaningful β values throughout the LER arcs based on the β and α values measured in the straight sections (where the phase advance/cell $\mu \neq 90°$) and the measured phase advance throughout the arcs. Implicit in this algorithm is the assumption that the lattice optics is "locally correct". The BPM amplitudes are used as a cross check for consistency. An alternative method interpolates phase and amplitude for the non-reading plane of the BPMs, thus circumventing the 90° problem.

2 BEAM OPTICS

2.1 Non-90°/cell optics

The equations have been derived in several other place sin the literature,[1, 2] we only cite the final result:

$$\beta_{2,m} = \frac{\cot\mu_{23,m} + \cot\mu_{12,m}}{\cot\mu_{23} + \cot\mu_{12}}\beta_2, \qquad (1)$$

$$\alpha_{2,m} = \frac{\cot\mu_{23,m} + \cot\mu_{12,m}}{\cot\mu_{23} + \cot\mu_{12}}\alpha_2 - \frac{\cot\mu_{23,m}\cot\mu_{12} - \cot\mu_{12,m}\cot\mu_{23}}{\cot\mu_{12} + \cot\mu_{23}}, \qquad (2)$$

where the index "$,m$" denotes measured values. This, of course, is what was worked out at CERN by P. Castro-Garcia to measure the lattice functions in LEP and what is also used at CESR and at the PEP-II High Energy Ring (HER),[3] although it is written here for the specific case calculating β, α at the middle one of three BPMs, see Fig. 1. It works quite well as long as the model phase advances μ between BPMs are suitable.

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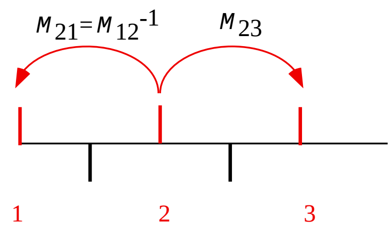

Figure 1: Location of BPM phase measurements

3 β FUNCTIONS IN 90° SECTIONS

When μ, the model phase advance/BPM for the section, is $(2n+1)\pi/2$, but more generally when $|\mu_{12} + \mu_{23}|$ is $n\pi$ $(n = 0, 1, \ldots)$ the above method fails. For this case, we quote the result from [1]:

$$\beta_{2,m} = \frac{\beta_1}{\beta_{1,m}}\frac{\sin^2\mu_{12}}{\sin^2\mu_{12,m}}\beta_2, \qquad (3)$$

$$\alpha_{2,m} = \cot\mu_{12,m} - \frac{\beta_{2,m}}{\beta_2}(\cot\mu_{12} - \alpha_2). \qquad (4)$$

Where the phase at BPM 2 is not known we can calculate it from $\beta_{1,m}, \alpha_{1,m}$:

$$\mu_{12,m} = \operatorname{arccot}\left(\frac{\beta_{1,m}}{\beta_1}(\cot\mu_{12} + \alpha_1) - \alpha_{1,m}\right) \qquad (5)$$

and thus we know $\beta_{2,m}$ as well. This is useful for both PEP-II rings since most BPMs are either x or y only.

4 ESTIMATE OF ERRORS

4.1 Uncertainty of the phase measurement

The BPM processors (RinQ)[4] derive phase and amplitude of the measured betatron oscillation from a combined sine and cosine fit:

$$a(n) = A\cos 2\pi\nu n + B\sin 2\pi\nu n, \qquad (6)$$

from which amplitude and phase are derived

$$a = \sqrt{A^2 + B^2}, \quad \phi = \arctan\frac{B}{A}, \qquad (7)$$

and are handed back to the central control computer. The rms residual for this fit, δa, is also returned. For an estimate of the accuracy in the phase ϕ we make the assumption that $\delta A = \delta B = \delta$ and have

$$\delta a = \sqrt{\frac{(A\delta)^2 + (B\delta)^2}{A^2 + B^2}} = \delta. \qquad (8)$$

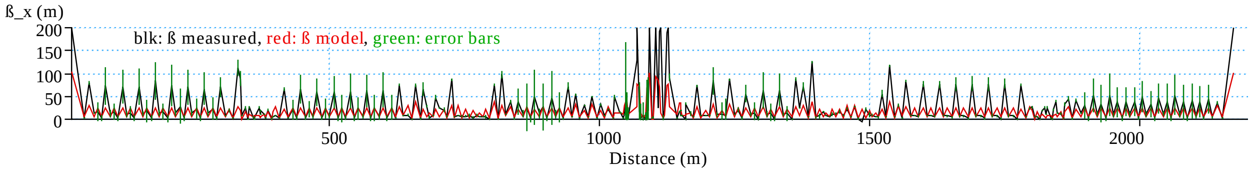

Figure 2: β_x in the LER at $\nu_x \approx 0.53$, before correcting β. Some error bars extending below 0 have been truncated.

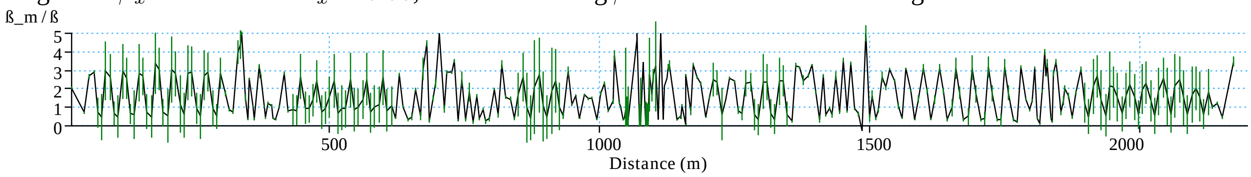

Figure 3: $\beta_{x,meas}/\beta_{x,model}$ for the above. Error bars in green

Error propagation gives the error in $\tan\phi$ as

$$\delta \tan\phi \quad = \quad \tan\phi \sqrt{\frac{\delta^2}{A^2} + \frac{\delta^2}{B^2}}. \tag{9}$$

We express A and B in terms of the available quantities a (the amplitude) and ϕ (the phase) using (7):

$$A = a\sin\phi, \quad B = a\cos\phi; \tag{10}$$

$$\delta\tan\phi = \delta \times \tan\phi \sqrt{\frac{1}{a^2\cos^2\phi} + \frac{1}{a^2\sin^2\phi}}, \tag{11}$$

and

$$\delta\phi = \frac{d\arctan(\tan\phi)}{d\tan\phi}\delta\tan\phi = \frac{\delta}{a}. \tag{12}$$

The uncertainty in the phase advance is

$$\delta\mu_{12,m} = \sqrt{(\delta\phi_1)^2 + (\delta\phi_2)^2} \tag{13}$$

4.2 *Uncertainty in the derived lattice functions*

For the local calculation in the non-90°-sections we have

$$\delta\beta_{2,m} \quad = \quad \frac{\sqrt{\frac{(\delta\mu_{12,m})^2}{\sin^4\mu_{12,m}} + \frac{(\delta\mu_{23,m})^2}{\sin^4\mu_{23,m}}}}{|\cot\mu_{12} + \cot\mu_{23}|}\beta_2 \tag{14}$$

$$\delta\alpha_{2,m} = \tag{15}$$

$$\frac{\sqrt{\frac{(\delta\mu_{12,m})^2}{\sin^4\mu_{12,m}}(\alpha_2^2+\cot^2\mu_{23}) + \frac{(\delta\mu_{23,m})^2}{\sin^4\mu_{23,m}}(\alpha_2^2+\cot^2\mu_{12})}}{|\cot\mu_{12}+\cot\mu_{23}|}.$$

Where $\beta_{2,m}$ is found using the measured $\beta_{1,m}$ and the phase advance $\mu_{12,m}$ from the previous location, we find

$$\delta\beta_{2,m} \quad = \quad \beta_1\beta_2\sin^2\mu_{12}\cdot \tag{16}$$

$$\sqrt{\frac{(\delta\beta_{1,m})^2}{\beta_{1,m}^4} + \frac{4\cot^2\mu_{12,m}}{\sin^4\mu_{12,m}}(\delta\mu_{12,m})^2},$$

$$\delta\alpha_{2,m} = \sqrt{\frac{(\delta\mu_{12,m})^2}{\sin^4\mu_{12,m}} + \frac{\cot\mu_{12} - \alpha_2}{\beta_2}(\delta\beta_{2,m})^2}. \tag{17}$$

For β at non-reading BPMs found using the phase advance derived from (5) one has to first substitute the phase into (3) so the derivatives w.r.t. $\beta_{1,m}$ and $\alpha_{1,m}$ can be found:

$$\delta\beta_{2,m} = \sqrt{\left(\frac{d\beta_{2,m}}{d\beta_{1,m}}\delta\beta_{1,m}\right)^2 + \left(\frac{d\beta_{2,m}}{d\alpha_{1,m}}\delta\alpha_{1,m}\right)^2}$$

$$= \frac{\beta_1\beta_2}{\beta_{1,m}}\sin^2\mu_{12}\sqrt{\delta\alpha_{1,m}^2 + \frac{\delta\beta_{1,m}^2}{\beta_{1,m}^2}(1-\alpha_{1,m})^2}. \tag{18}$$

5 FORWARD & BACKWARD CALCULATION

As in Ref [1] we average forward and backward calculation wherever Eq. (3) is used, except that we now combine the uncertainties of β_m in the calculation: The overall normalized amplitude, is calculated by

$$A_{norm} = \frac{\sum_{BPMs}\frac{A_i}{\sqrt{\beta_i}}w_i}{\sum_{BPMs}w_i}, \tag{19}$$

$$w_i = \frac{\beta_i}{\delta^2}, \tag{20}$$

using the model values for β_i. We then average backward and forward values using weights

$$p_{f,b} = \frac{1}{(a_{f,b}/\sqrt{\beta_{f,b,m}} - A_{norm})^2(\delta\beta_{f,b,m})^2}; \tag{21}$$

the indices f, b referring to forward and backward-calculation, resp. In cases where the amplitude is not available (bad BPM or BPM reading only in the other plane) the weights simplify to

$$p_i = \frac{1}{(\delta\beta_{i,m})^2}. \tag{22}$$

There may be some cases where the $\beta_{i,m}$ values calculated in either forward or backward direction are not meaningful (*e.g.* negative due to an error in BPM phase) in which case only the valid reading and its error are used.

The error on the averaged result can in principle be calculated in two ways: Either one treats the two β values as two independent samples (which they are since they are derived from different BPMs) of the same population, in

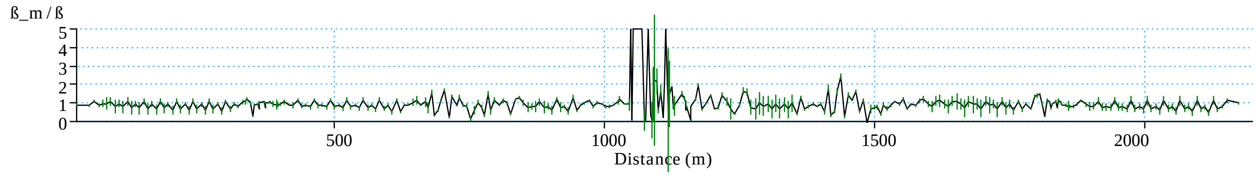

Figure 4: $\beta_{meas}/\beta_{model}$ in the LER at low ν_x, after correction. Some large spikes have been removed. Error bars in green

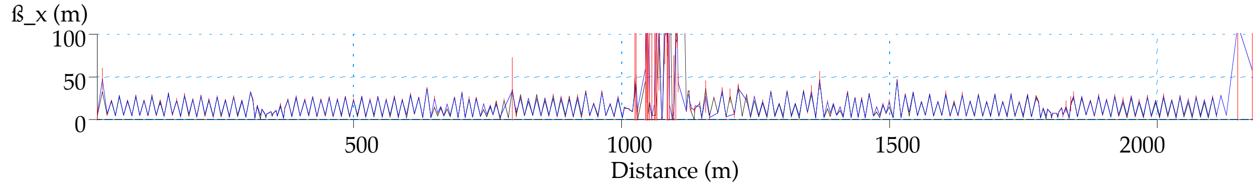

Figure 5: Horizontal β calculated using the alternate method (blue) and the method of sec. 3 (black). Error bars are in red.

which case the error of the average is just the standard deviation, or one takes the average β value as just a derived quantity in which case normal error propagation should be used. Here we take the latter approach since a statistical analysis based on just two samples appears questionable. The error of the average β is then

$$\delta\beta_m = \sqrt{\frac{(\delta\beta_{f,m})^2 p_f + (\delta\beta_{b,m})^2 p_b}{p_f + p_b}} \tag{23}$$

Note that the error contributions are also weighted.

6 ALTERNATE METHOD

Another method to avoid the 90° problem is to interpolate the phase and amplitude measurements between the measured points, in our case to the non-reading plane of the BPMs and then employing the regular formula to these readings, which are about 45° apart, as well. This interpolation again requires use of the unperturbed lattice functions and, in addition, assumes that the calibration of the BPMs is good.

The code that employs this method calculates the beta function at a BPM three ways, as the first BPM in a group of three, as the middle member of a different group and as the last member. The three measurements are weighted according to the product of the sine functions (phase between BPMs) involved.

In the 90° arcs, and in all straight sections other than the highly coupled interaction region straight, these three measurements are in good agreement. In the interaction region straight we often find disagreement between those measurements and if the spread between the three measurements is too great the amplitude of the oscillation is used to calculate the beta function. The calibration of amplitude versus $\sqrt{\beta}$ is made using "good" beta calculations at BPMs preceding this "bad" measurement.

If the horizontal motions at positions 1 and 3 are given by (n is the turn number)

$$x_1 = a_1\cos(2\pi\nu n + \phi_1); \quad x_3 = a_3\cos(2\pi\nu n + \phi_3) \tag{24}$$

we get at position 2 in between

$$x_2 = m_{11}^{12} a_1 \cos(2\pi\nu n + \phi_1) \tag{25}$$

$$+\frac{m_{12}^{12}}{m_{12}^{13}}(a_3\cos(2\pi\nu n + \phi_3) - m_{11}^{13} a_1 \cos(2\pi\nu n + \phi_1)).$$

Defining A_1 and A_3 so that

$$x_2 = A_1\cos(2\pi\nu n + \phi_1) + A_3\cos(2\pi\nu n + \phi_3) \tag{26}$$

and denoting m_{mn}^{ij} as the m_{mn} element of the transport matrix from i to j we find

$$A_1 = \left(m_{11}^{12} - \frac{m_{12}^{12}}{m_{12}^{13}} m_{11}^{13}\right) a_1, \tag{27}$$

$$A_3 = \frac{m_{12}^{12}}{m_{12}^{13}} a_3 \tag{28}$$

and

$$x_2 = a_2\cos(2\pi\nu n + \phi_2), \tag{29}$$

where:

$$a_2 = \sqrt{A_1^2 + A_3^2 + 2A_1A_3\cos(\phi_3 - \phi_1)}, \tag{30}$$

$$\phi_2 = \phi_1 + \arcsin\left[\frac{A_3}{a_2}\sin(\phi_3 - \phi_1)\right]. \tag{31}$$

7 MEASUREMENTS

In Fig. 2 the measured β is shown together with the model, In Fig. 3 the ratio of measured to model beta. The β beating apparent in the example was observed when the working point in the LER was moved to 0.53 in x. This was subsequently fixed by tweaking quadrupole strengths, the result is shown in Fig. 4. Note that in the latter case the error bars are substantially smaller than in the former. In Fig. 5 we show the result using the formalism of sec. 6 together with that using the formalism of sec. 3.

8 REFERENCES

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