TIME EVOLUTION OF THE PARTICLE DISTRIBUTION IN BEAM PHYSICS: THE ECHO EFFECT

G. Bassi, DESY, Hamburg, Germany and Dept. of Physics University of Bologna, ITALY
H. Mais, DESY, Hamburg, Germany
A. Bazzani, G. Turchetti, INFN and Dept. of Physics University of Bologna, ITALY

Abstract

The understanding of the time evolution of particle distributions in accelerators is an important problem in beam physics. In this contribution we study some numerical aspects related to this question. As a benchmark problem we consider the beam echo in proton storage rings.

1 INTRODUCTION

A good understanding of the time evolution of particle distributions in storage rings is an important problem in beam physics, especially if one is interested in the influence of noise in problems such as coasting beams (diffusion out of stable rf-buckets) or echo-diffusion in proton machines. The beam echo has been studied extensively by Stupakov et al [1], [2], and we will consider it as a benchmark problem for our numerical investigations.

2 THE MODEL

Echoes in particle beams are based on the sensitive link between macroscopically measurable quantities (motion of the centroid of the beam) and the microscopic (Hamiltonian) phase space dynamics of the particles. A simple mathematical model, which can be solved analitycally was given by Stupakov

$$\begin{aligned} \dot{x} &= \omega(J)p \\ \dot{p} &= -\omega(J)x. \end{aligned} \quad (1)$$

where the tune $\omega(J)$ is supposed to depend linearly on the amplitude of the oscillations

$$\omega(J) = \omega_0 + \omega_1 J. \qquad J = \frac{x^2+p^2}{2}.$$

The phase space flow reads

$$\begin{aligned} x =& \quad x_0 cos(\omega(J)t) + p_0 sin(\omega(J)t) \\ p =& \ -x_0 sin(\omega(J)t) + p_0 cos(\omega(J)t). \end{aligned} \quad (2)$$

Assuming that the initial distribution function is subject to a dipole kick d

$$\rho(x,p) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+(p-d)^2}{2\sigma^2}}$$

and following the calculations of Stupakov and Kauffmann it is possible to derive an analytical formula that describes the evolution of the centroid $< x >$ (first moment). Due to the non linear character of this model $< x >$ relaxes towards zero in an exponential way. However, if we apply a quadrupole kick q at time τ

$$\begin{aligned} x' &= x \\ p' &= p - qx \end{aligned} \quad (3)$$

even if the filamentation phenomenon smears out the distribution in the phase space the coherence of the phase dynamics induces an echo effect in the $< x >$ value which can be observed at time 2τ. When the quadrupole kick q is small the phenomenon reveals a regular echo train subsequent to the first echo signal at times $t = 4\tau, 6\tau$, etc. In Fig. 1 and Fig 2 (left) we have calculated the centroid motion applying the Liouville theorem backward in time

$$< x >= \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} x\rho(x_{-t}(x_0,p_0), p_{-t}(x_0,p_0))dx$$

for the parameters $\omega_0 = 1.2885, \omega_1 = 0.186, q = 0.04, d = 1, \sigma = 0.2, \tau = 200$.

If one wants to study more realistic cases (longitudinal echoes, transverse echoes with beam-beam interaction) and if one wants to include the effects of noise one has to perform numerical studies.

One way is to study the stochastic dynamical system directly. For example we have solved the equations for the system (1) perturbed by an additive Gaussian white ($< \xi(t)\xi(t') >= 2\delta(t-t')$) noise

$$\begin{aligned} \dot{x} &= \omega(J)p \\ \dot{p} &= -\omega(J)x + \sqrt{D}\xi(t) \end{aligned} \quad (4)$$

using a symplectic integrator of the second order [3].

In Fig. 2 we have plotted the decay of the echo as a function of the noise strenght for the parameters $\omega_0 = 4.2885, \omega_1 = 2.186, q = 0.08, d = 1, \tau = 20s$.

An alternative way is to solve the corresponding Fokker-Planck equation, a partial differential equation for the probability to find the system at time t between $(x, x+dx)$ and $(p, p+dp)$. For the system (4) it reads

$$\frac{\partial\rho}{\partial t} = -p\frac{\partial}{\partial x}\omega(J)\rho + x\frac{\partial}{\partial p}\omega(J)\rho + D\frac{\partial^2\rho}{\partial p^2}$$

Due to the sensitive dependence of the first moment on the microscopic dynamics (filamentation in phase space, see

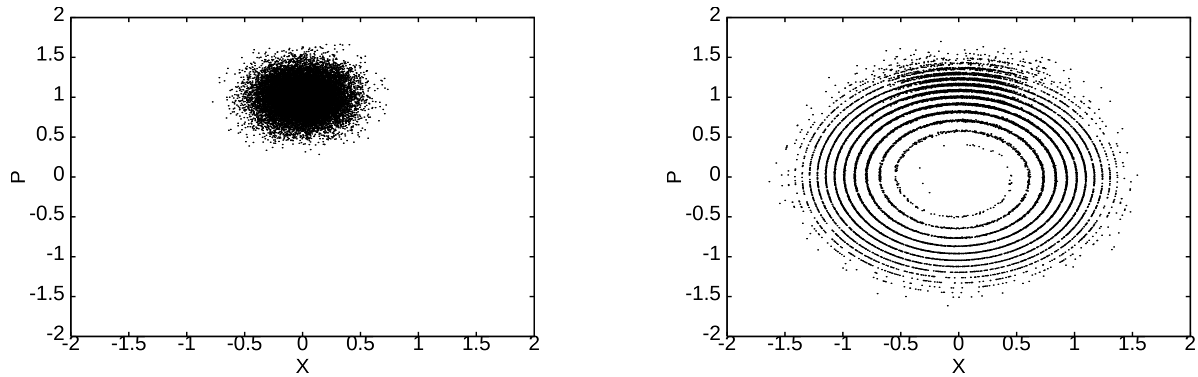

Figure 1: Left: initial distribution. Right: distribution at time τ.

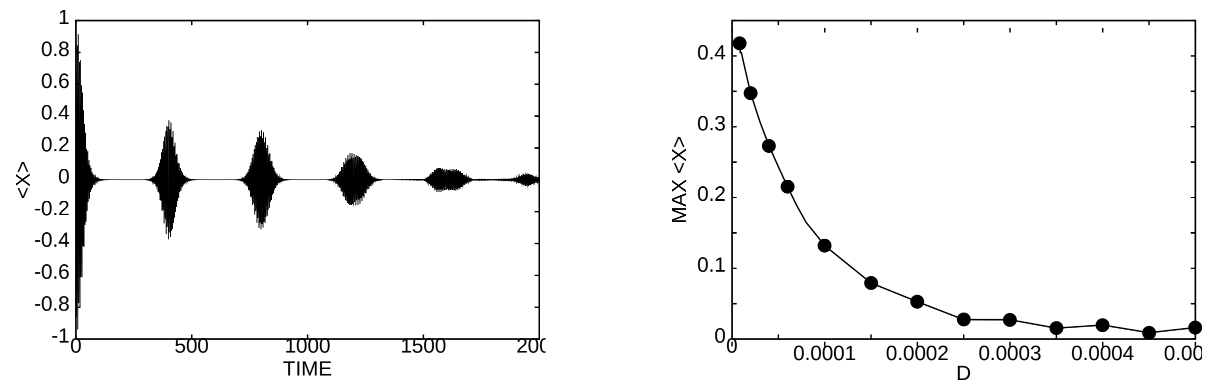

Figure 2: Left: time evolution of $< x >$. Right: Max amplitude of $< x >$ versus the diffusion D.

Fig. 1) one can expect that the numerics is highly non trivial. The reliability of ordinary numerical schemes has to be checked very carefully by comparision with multiparticle simulations or by comparing the noiseless (Liouville) case with the exact solutions.

Numerical schemes based on finite difference methods and operator splitting tecnhiques have been described in [4] and [5]. In Fig. 3, 4, 5 we show some results which were obtained with these schemes in the deterministic (noiseless limit) case using a cartesian coordinate system. The code developed in [4] is not able to reproduce the echo while the code developed in [5] gives an accurate value for the echo, although the error in the distribution is quite large (see Fig. 4). From the properties of the deterministic dynamics one can understand that a numerical scheme must pay attention to the fine structure of the distribution in the radial direction (in the angular direction the distribution translates rigidly). The method developed in [5] gives very good results (see Fig 4), when one uses a grid of polar coordinates with

$$\Delta r = 0.006, \Delta\theta = \frac{2\pi}{400}.$$

3 CONCLUSIONS

The Fokker-Planck equation is an important tool to study the time evolution of the particle density in accelerators, especially if one wants to investigate the dynamics of the tails in the distribution. Accurate and reliable solvers are needed to solve Hamiltonian systems plus weak noise. The beam echo, used as a benchmark problem in this contribution, demonstrates the problems one can have with numerical solvers: the finite grid structure and the used interpolation procedure can easily smear out the dynamics in such a way that the echo disappears. So some care has to be taken, when one applies the Fokker-Planck approach to Hamiltonian problems plus weak noise. Multigrid schemes, or coordinate systems, which are better adjusted to the dynamics could be developed. Progress in this direction is underway.

4 REFERENCES

[1] G.Stupakov, “Echo Effect in Hadron Colliders”, SSCL Report 579 (1992).

[2] G.Stupakov and S. Kauffmann, “Echo Effect in Accelerators”, SSCL Report 587 (1992).

[3] M. Seeßelberg et al, Z. Phys. C62:63, 1994

[4] M.P. Zorzano, Ph. D. Thesis “Numerical integration of the Fokker-Planck Equation and Application to Stochastic Beam Dynamics in Storage Rings”, Universidad Complutense de Madrid-DESY, March 1999.

[5] R. L. Warnock, “A General Method For Propagation Of The Phase Space Distribution, With Application To The Sawtooth Instability”,SLAC-PUB-8404 - Stanford, CA:SLAC, 15 Mar 2000.

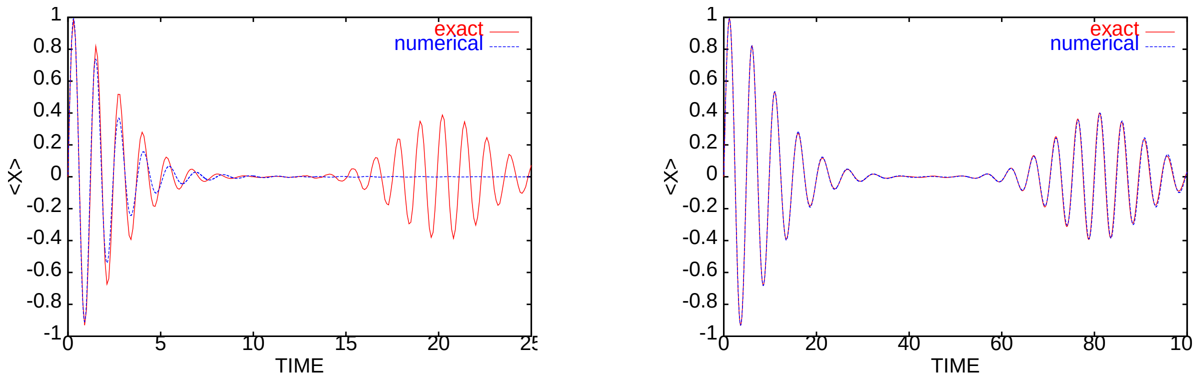

Figure 3: Left: evolution of $< x >$ with the scheme developed in [4] in cartesian coordinates for $\omega_0 = 4, \omega_1 = 2, q = 0.08, d = 1, \tau = 10s, \Delta t = 0.0005, \Delta x = 0.006$. Right: evolution of $< x >$ with the scheme developed in [5] in cartesian coordinates for $\omega_0 = 1, \omega_1 = 0.5, q = 0.08, d = 1, \tau = 40s, \Delta t = 0.01, \Delta x = 0.006$.

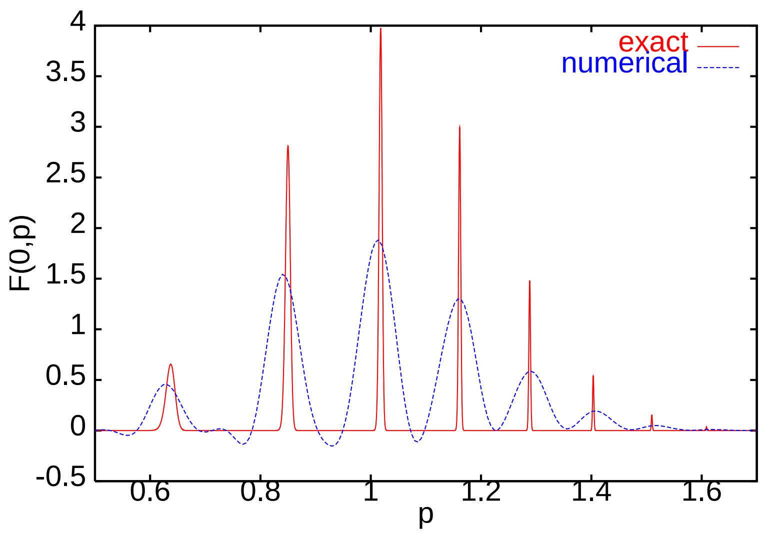

Figure 4: Comparison beetween the scheme developed in [5] and the exact solution in cartesian coordinates at $t = 81s$. $\omega_0 = 1, \omega_1 = 0.5, q = 0.08, d = 1, \tau = 40s, \Delta t = 0.01, \Delta x = 0.006$.

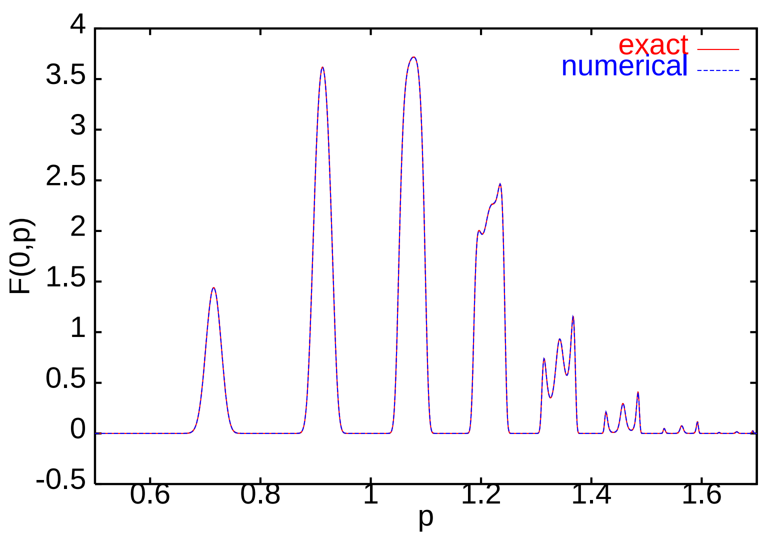

Figure 5: Comparison beetween the scheme developed in [5] and the exact solution in polar coordinates at $t = 79s$. $\omega_0 = 1, \omega_1 = 0.5, q = 0.08, d = 1, \tau = 40s, \Delta t = 0.01, \Delta r = 0.003, \Delta\theta = \frac{2\pi}{400}$.