SOLITARY WAVES IN AN INTENSE BEAM PROPAGATING THROUGH A SMOOTH FOCUSING FIELD

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Abstract

Based on the Vlasov-Maxwell equations describing the self-consistent nonlinear beam dynamics and collective processes, the evolution of an intense sheet beam propagating through a periodic focusing field has been studied. In an earlier paper [1] it has been shown that in the case of a beam with uniform phase space density the Vlasov-Maxwell equations can be replaced exactly by the macroscopic warm fluid-Maxwell equations with a triple adiabatic pressure law. In this paper we demonstrate that starting from the macroscopic fluid-Maxwell equations a nonlinear Schroedinger equation for the slowly varying wave amplitude (or a set of coupled nonlinear Schroedinger equations for the wave amplitudes in the case of multi-wave interactions) can be derived. Properties of the nonlinear Schroedinger equation are discussed, together with soliton formation in intense particle beams.

1 INTRODUCTION

Of particular importance in modern accelerators and storage rings operating at high beam currents and charge densities are the effects of the intense self-fields produced by the beam space charge and current on determining detailed equilibrium, stability and transport properties. In general, a complete description of collective processes in intense charged particle beams is provided by the Vlasov-Maxwell equations for the self-consistent evolution of the beam distribution function and the electromagnetic fields. As shown in [1] in the case of a sheet beam with constant phase-space density the Vlasov-Maxwell equations are fully equivalent to a warm-fluid model with zero heat flow and triple-adiabatic equation-of-state.

In the present paper we demonstrate that starting from the hydrodynamic equations, and using the renormalization group (RG) technique [2, 3, 4, 5] a nonlinear Schroedinger equation for the slowly varying single-wave amplitude can be derived. The renormalized solution for the beam density describes the process of formation of periodic *holes* in intense particle beams.

2 THE HYDRODYNAMIC MODEL

We begin with the hydrodynamic model derived in [1]

$$\frac{\partial \varrho}{\partial s} + \frac{\partial}{\partial x}(\varrho v) = 0,$$

$$\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial x} + v_T^2 \frac{\partial}{\partial x}\varrho^2 = -G(s)x - \frac{\partial \psi}{\partial x}, \qquad (2.1)$$

$$\frac{\partial^2 \psi}{\partial x^2} = -2\pi K \varrho.$$

Here $\varrho(x;s) = n(x;s)/N$ and $v(x;s)$ are the normalized density and the current velocity, respectively, $G(s + S) = G(s)$ is the periodic focusing lattice coefficient, $v_T^2 = 3\widehat{P}_0/2\widehat{n}_0^3$ is the normalized thermal speed-squared, and $\widehat{P}_0/\widehat{n}_0^3 = N^2/12A^2$ is a constant coefficient [1], where N is the area density of sheet beam particles, and A is the constant phase-space density. Moreover, $\psi(x;s)$ is the normalized self-field potential

$$\psi(x;s) = \frac{e_b \phi(x;s)}{m_b \gamma_b \beta_b^2 c^2},$$

where $\phi(x;s)$ is the electrostatic (space-charge) potential, m_b and e_b are the rest mass and charge of a beam particle, and β_b and γ_b are the relative particle velocity and Lorentz factor, respectively. Finally, the quantity K is the normalized self-field perveance defined by

$$K = \frac{2Ne_b^2}{m_b \gamma_b^3 \beta_b^2 c^2}.$$

In what follows the analysis is restricted to the smooth focusing approximation

$$G(s) = G = \mathrm{const}, \qquad (2.2)$$

and assume that there exist nontrivial stationary solutions to (2.1) in the interval $x \in \left(-x^{(-)}, x^{(+)}\right)$, and that the sheet beam density is zero $(\varrho = 0)$ outside of the interval. The change of variables

$$\xi = x + x^{(-)}, \qquad \Psi = \psi - Gx^{(-)}x \qquad (2.3)$$

enables us to rewrite (2.1) in the form

$$\frac{\partial \varrho}{\partial s} + \frac{\partial}{\partial \xi}(\varrho v) = 0,$$

$$\frac{\partial v}{\partial s} + v\frac{\partial v}{\partial \xi} + v_T^2 \frac{\partial}{\partial \xi}\varrho^2 = -G\xi - \frac{\partial \Psi}{\partial \xi}, \qquad (2.4)$$

$$\frac{\partial^2 \Psi}{\partial \xi^2} = -2\pi K \varrho.$$

Clearly, the system (2.4) possesses a stationary solution

$$\varrho_0 = \frac{G}{2\pi K}, \qquad v_0 \equiv 0, \qquad \Psi_0 = -\frac{G\xi^2}{2} + \mathrm{const}. \qquad (2.5)$$

Here, the uniform density ϱ_0 is normalized according to

$$x^{(-)} + x^{(+)} = \frac{1}{\varrho_0} = \frac{2\pi K}{G}. \qquad (2.6)$$

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3 RENORMALIZATION GROUP REDUCTION OF THE HYDRODYNAMIC EQUATIONS

Following the basic idea of the RG method, we represent the solution to equations (2.4) in the form of a standard perturbation expansion in a formal small parameter ϵ as

$$\varrho = \varrho_0 + \sum_{k=1}^{\infty} \epsilon^k \varrho_k, \qquad v = \sum_{k=1}^{\infty} \epsilon^k v_k, \qquad (3.1)$$

$$\Psi = -\frac{G\xi^2}{2} + \sum_{k=1}^{\infty} \epsilon^k \Psi_k.$$

Before proceeding with explicit calculations order by order, we note that in all orders the perturbation equations acquire similar general form. Eliminating v_n and Ψ_n, it is possible to obtain a single equation for ϱ_n alone, i.e.,

$$\frac{\partial^2 \varrho_n}{\partial s^2} - 2\varrho_0^2 v_T^2 \frac{\partial^2 \varrho_n}{\partial \xi^2} + G\varrho_n = \frac{\partial \alpha_n}{\partial s} - \varrho_0 \frac{\partial \beta_n}{\partial \xi}, \qquad (3.2)$$

where the functions $\alpha_n(\xi; s)$ and $\beta_n(\xi; s)$ involve contributions from previous orders and are considered known. It is evident that in first order $\alpha_1 = \beta_1 = 0$. Imposing the condition

$$\int\limits_0^{1/\varrho_0} \mathrm{d}\xi \varrho_1(\xi; s) = 0, \qquad (3.3)$$

which means that linear perturbation to the uniform stationary density ϱ_0 should average to zero and not affect the normalization properties on the interval $\left(0, x^{(-)} + x^{(+)}\right)$, we obtain the first-order solution

$$\varrho_1(\xi; s) = \sum_{m \neq 0} \mathcal{A}_m e^{i\chi_m(\xi;s)}, \quad \chi_m(\xi; s) = \omega_m s + m\sigma\xi. \qquad (3.4)$$

Here, $\mathcal{A}_m$ are constant complex wave amplitudes, and the following conventions and notations

$$\omega_{-m} = -\omega_m, \qquad \sigma = \frac{G}{K}, \qquad \mathcal{A}_{-m} = \mathcal{A}_m^*. \qquad (3.5)$$

have been introduced. Moreover, the discrete mode frequencies ω_m are determined from the dispersion relation

$$\omega_m^2 = G + \frac{v_T^2 \sigma^4}{2\pi^2} m^2. \qquad (3.6)$$

In addition, the first-order solution for the current velocity can be expressed as

$$v_1(\xi; s) = -\frac{1}{\varrho_0 \sigma} \sum_{m \neq 0} \frac{\omega_m}{m} \mathcal{A}_m e^{i\chi_m(\xi;s)}, \qquad (3.7)$$

In obtaining the second-order perturbation equation (3.2), we note that

$$\alpha_2 = -\frac{\partial}{\partial \xi}(\varrho_1 v_1), \qquad \beta_2 = -\frac{1}{2}\frac{\partial}{\partial \xi}\left(v_1^2 + 2v_T^2 \varrho_1^2\right). \qquad (3.8)$$

Thus the second-order solution for the density $\varrho_2(\xi; s)$ is found to be

$$\rho_2(\xi; s) = -\sum_{m,k \neq 0} \alpha_{mk} \mathcal{A}_m \mathcal{A}_k e^{i[\chi_m(\xi;s) + \chi_k(\xi;s)]}, \qquad (3.9)$$

where

$$\alpha_{mk} = \frac{m+k}{\mathcal{D}_{mk}} \left[\frac{\omega_k(\omega_m + \omega_k)}{k\varrho_0} + \frac{m+k}{2\varrho_0} \left(\frac{v_T^2 \sigma^4}{2\pi^2} + \frac{\omega_m \omega_k}{mk} \right) \right], \qquad (3.10)$$

$$\mathcal{D}_{mk} = -(\omega_m + \omega_k)^2 + \frac{v_T^2 \sigma^4}{2\pi^2}(m+k)^2 + G. \qquad (3.11)$$

Having determined ϱ_2, the second-order current velocity $v_2(\xi; s)$ can be found in a straightforward manner. The result is

$$v_2(\xi; s) = \frac{1}{\varrho_0 \sigma} \sum_{m,k \neq 0} \beta_{mk} \mathcal{A}_m \mathcal{A}_k e^{i[\chi_m(\xi;s) + \chi_k(\xi;s)]}, \qquad (3.12)$$

where

$$\beta_{mk} = \frac{\omega_k}{k\varrho_0} + \frac{\omega_m + \omega_k}{m+k} \alpha_{mk}, \qquad \beta_{m,-m} = 0. \qquad (3.13)$$

In third order, the functions α_3 and β_3 entering the right-hand-side of equation (3.2) can be calculated utilizing the already determined quantities from the first and second orders, according to

$$\alpha_3 = -\frac{\partial}{\partial \xi}(\varrho_1 v_2 + \varrho_2 v_1), \qquad (3.14)$$

$$\beta_3 = -\frac{\partial}{\partial \xi}\left(v_1 v_2 + 2v_T^2 \varrho_1 \varrho_2\right). \qquad (3.15)$$

It is important to note that the right-hand-side of equation (3.2) for ϱ_3 contains terms which yield oscillating terms with constant amplitudes to the solution for ϱ_3. Apart from these, there is a resonant term (proportional to $e^{i\chi_m(\xi;s)}$) leading to a secular contribution. To complete the renormalization group reduction of the hydrodynamic equations, we select this particular resonant third-order term on the right-hand-side of equation (3.2). The latter can be written as

$$\left(\frac{\partial \alpha_3}{\partial s} - \varrho_0 \frac{\partial \beta_3}{\partial \xi} \right)_{res} = \sum_{m,k \neq 0} \Gamma_{mk} \mathcal{A}_m |\mathcal{A}_k|^2 e^{i\chi_m(\xi;s)}, \qquad (3.16)$$

where

$$\Gamma_{mk} = \frac{m}{\varrho_0} \left[\omega_m \left(\beta_{mk} + \frac{\omega_k \alpha_{mk}}{k} \right) + \frac{m\omega_k \beta_{mk}}{k} + \frac{v_T^2 \sigma^4}{2\pi^2} m\alpha_{mk} \right]. \qquad (3.17)$$

Some straightforward algebra yields the solution for $\varrho_3(\xi; s)$ to equation (3.2) in the form

$$\varrho_3(\xi; s) = \sum_{m \neq 0} \mathcal{P}_m(\xi; s) e^{i\chi_m(\xi;s)} + \dots, \qquad (3.18)$$

where the dots stand for non-secular oscillating terms. Moreover, the amplitude $\mathcal{P}_m(\xi;s)$ is secular and satisfies the equation

$$\widehat{\mathcal{L}}_m(\xi;s)\mathcal{P}_m(\xi;s) = \sum_{k\neq 0} \Gamma_{mk}\mathcal{A}_m|\mathcal{A}_k|^2, \tag{3.19}$$

where the operator $\widehat{\mathcal{L}}_m$ is defined by

$$\widehat{\mathcal{L}}_m = \frac{\partial^2}{\partial s^2} + 2i\left(\omega_m \frac{\partial}{\partial s} - \frac{v_T^2\sigma^3}{2\pi^2} m \frac{\partial}{\partial \xi}\right) - \frac{v_T^2\sigma^2}{2\pi^2}\frac{\partial^2}{\partial \xi^2}. \tag{3.20}$$

We can now construct the perturbative solution for ϱ up to third order in the small parameter ϵ. Confining attention to the constant stationary density ϱ_0 and the fundamental modes (first harmonic in the phase χ_m), we obtain

$$\varrho(\xi;s) = \varrho_0 + \epsilon \sum_{m\neq 0} \left[\mathcal{A}_m + \epsilon^2 \mathcal{P}_m(\xi;s)\right] e^{i\chi_m(\xi;s)}. \tag{3.21}$$

Following the basic philosophy of the RG method, we introduce the intermediate coordinate X and "time" S and transform equation (3.21) to

$$\varrho(\xi;s) = \varrho_0 + \epsilon \sum_{m\neq 0} \{\mathcal{A}_m(X;S)$$
$$+\epsilon^2[\mathcal{P}_m(\xi;s) - \mathcal{P}_m(X;S)]\} e^{i\chi_m(\xi;s)}. \tag{3.22}$$

Note that the transition from equation (3.21) to equation (3.22) can always be performed by enforcing the constant amplitude $\mathcal{A}_m$ to be dependent on X and S, which is in fact the procedure for renormalizing the standard perturbation result. Since the general solution for $\varrho(\xi;s)$ should not depend on X and S, by applying the operator $\widehat{\mathcal{L}}_m(X;S)$ [which is the same as that in equation (3.20) but with $\xi \to X$ and $s \to S$] on both sides of equation (3.22), we obtain

$$\widehat{\mathcal{L}}_m(X;S)\mathcal{A}_m(X;S) = \sum_{k\neq 0} \Gamma_{mk}\mathcal{A}_m(X;S)|\mathcal{A}_k(X;S)|^2, \tag{3.23}$$

where we have dropped the formal parameter ϵ on the right-hand-side. Since the above equation should hold true for any choice of X and S, we can set $X = \xi$ and $S = s$. Thus, we obtain the so-called proto RG equation [3, 4, 5]

$$\widehat{\mathcal{L}}_m(\xi;s)\mathcal{A}_m(\xi;s) = \sum_{k\neq 0} \Gamma_{mk}\mathcal{A}_m(\xi;s)|\mathcal{A}_k(\xi;s)|^2. \tag{3.24}$$

Introducing the new variable

$$\zeta_m = \frac{v_T^2\sigma^3 m}{2\pi^2} s + \omega_m \xi \tag{3.25}$$

and neglecting the second order derivatives $\partial^2/\partial s^2$ and $\partial^2/\partial s\partial\zeta_m$, we finally arrive at the RG equation for the m-th mode amplitude

$$2i\omega_m \frac{\partial \mathcal{A}_m}{\partial s} - \frac{v_T^2\sigma^2 G}{2\pi^2}\frac{\partial^2 \mathcal{A}_m}{\partial \zeta_m^2} = \sum_{k\neq 0} \Gamma_{mk}\mathcal{A}_m|\mathcal{A}_k|^2. \tag{3.26}$$

4 THE NONLINEAR SCHROEDINGER EQUATION FOR A SINGLE MODE

Equation (3.26) represents a system of coupled nonlinear Schroedinger equations for the mode amplitudes. Neglecting the contribution from modes with $k \neq m$, for a single mode amplitude $\mathcal{A}_m$, we obtain the equation

$$2i\omega_m \frac{\partial \mathcal{A}_m}{\partial s} - \frac{v_T^2\sigma^2 G}{2\pi^2}\frac{\partial^2 \mathcal{A}_m}{\partial \zeta_m^2} = -\Gamma_m|\mathcal{A}_m|^2\mathcal{A}_m, \tag{4.1}$$

where

$$\Gamma_m = -\Gamma_{mm} = \frac{2}{3G\varrho_0^2}\left(16\omega_m^4 - 11G\omega_m^2 + G^2\right). \tag{4.2}$$

It is easy to verify that Γ_m is always positive. In nonlinear optics equation (4.1) is known to describe the formation and evolution of the so-called *dark solitons* [6]. In the case of charged particle beams these correspond to the formation of *holes* or *cavitons* in the beam. Since the renormalized solution for the beam density $\varrho(\xi;s)$ can be expressed as

$$\varrho(\xi;s) = \varrho_0 + \sum_{m\neq 0} \mathcal{A}_m(\xi;s) e^{i\chi_m(\xi;s)}. \tag{4.3}$$

these holes have periodic structure in space ξ and "time" s.

5 CONCLUDING REMARKS

Based on the renormalization group method, a system of coupled nonlinear Schroedinger equations has been derived for the slowly varying amplitudes of interacting beam-density waves. Under the approximation of an isolated wave neglecting the effect of the rest of the waves, this system reduces to a single nonlinear Schroedinger equation with repulsive nonlinearity. The latter describes the formation and evolution of holes in intense charged particle beams.

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