QUANTUM-LIKE CORRECTIONS
AND TOMOGRAPHY IN BEAM PHYSICS

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Abstract

A novel tomographic approach to charged-particle beam physics is suggested in the framework of Thermal Wave Model (TWM). It is shown that the particle beam transport in the phase-space can be described in terms of a marginal distribution which has the features of a classical probability distribution, including its positive definiteness. It is shown that the above marginal distribution satisfies a Fokker-Planck-like equation as well as contains all the information of the Wigner function, even if the latter is not positive definite. Nevertheless, one can directly start from the classical single-particle physics where the potential is given and go directly to the Fokker-Planck-like equation which incorporates all the quantum-like effects of TWM.

1 INTRODUCTION

In accelerator physics, the main tool of the conventional approaches to describe the particle-beam behavior is represented by the positive classical probability-distribution function of the particle in the phase space and its evolution along the beam path [1]. On the other hand, nonconventional quantum-like methodologies have been recently proposed [2]. In particular, the thermal wave model (TWM) [3] has been constructed by making a *transition* from the *geometric electron optics* [4] to a *wave electron optics* in order to describe the behavior of a particle beam in terms of a complex function satisfying the Schrödinger-like equation where the beam emittance plays the role of the Planck's constant. TWM takes into account the electronic rays spreading due to finite temperature and its nature is completely classical. Nevertheless, the formalism used is mathematically equivalent to the quantum one. This procedure to transit from the classical to a quantum-like formalism (but still describing a classical system), is fully similar to the one of Gloge and Marcuse [5] to recover e.m. wave optics from e.m. geometrical optics to construct the quantum-like theory of light rays in paraxial approximation. In semiclassical approximation, the above transition can be obtained also with a *deformation procedure* to the classical phase-space equation for electronic rays and for an arbitrary potential [6]. But, whatever the quantum-like procedure used is, to perform the transition, the following considerations hold.

- The classical 2-D phase-space distribution, say $\rho(x,p,z)$, for the transverse beam dynamics (x and p being the single-particle position and momentum, respectively, and z being the propagation coordinate which plays the role of a time-like variable), is solution of the following equation:

$$\left\{\frac{\partial}{\partial z} + p\frac{\partial}{\partial x} - \left(\frac{\partial U}{\partial x}\right)\frac{\partial}{\partial p}\right\}\rho = 0 \quad , \tag{1}$$

which describes a phase-space evolution of electronic rays, and where $U = U(x,z)$ is an effective dimensionless potential acting on each single particle. By introducing the operator

$$\widehat{\mathcal{D}U}\left(\widehat{A},\widehat{B},z\right) \equiv U\left(\widehat{A}+\widehat{B},z\right) - U\left(\widehat{A}-\widehat{B},z\right) \quad , \tag{2}$$

for any operators $\widehat{A}$ and $\widehat{B}$, the above transition allows us to go from classical to a quantum-like phase-space distribution, say $\rho_w(x,p,z)$ satisfying the following von Neumann-Moyal-like equation [7]

$$\left\{\frac{\partial}{\partial z} + p\frac{\partial}{\partial x} + \frac{i}{\epsilon}\widehat{\mathcal{D}U}\left(x, i\frac{\epsilon}{2}\frac{\partial}{\partial p}, z\right)\right\}\rho_w = 0 \quad . \tag{3}$$

- Within the classical context, ρ is positive definite whilst ρ_w is not and coincides with a Wigner-like function. In fact, due to a quantum-like uncertainty relation introduced by the thermal spreading among the electronic rays, ρ_w can be also negative and is appropriately called (as in quantum mechanics) a *quasidistribution*. The above deformation procedure has shown that, in semiclassical approximation, (1) and (3) formally coincide unless than phase-space regions of size smaller than ϵ, where, due to the thermal uncertainty, one cannot resolve among two or more electronic rays [6]. Correspondingly, ρ and ρ_w, within the same above hypothesis, seem to be equivalent. Additionally, other aspects of the quantum-like descriptions given by the Wigner-like function and Husimi function [8] were considered in the framework of TWM [6, 7, 9].

- When the semiclassical approximation is removed, thus ρ and ρ_w are not equivalent anymore, and the discrepancy (quantum-like corrections) between these two distributions represents a quantum-like effect that in TWM accounts for an effective description of charged-particle beams in the presence of non-negligible thermal spreading among the electronic rays [6].

From the above considerations, we can conclude that TWM, through the quantization procedure, provides for a map

$$\rho\,(x,\,p,\,z) \Longrightarrow \rho_w\,(x,\,p,\,z)\ \ ,$$

like in quantum mechanics. In order to keep the above effective quantum-like description in a classical context, we now ask: *is it possible to construct an invertible map which allows us to reverse the above transition and go back from ρ_w to a classical distribution but containing all the information of ρ_w?* In the next section we try to give a positive answer to this question, mainly on the basis of investigations that, recently, introduced a new method for measuring quantum states in quantum mechanics and quantum optics, called optical tomography [10, 11] or symplectic tomography [12]. As was shown in [13, 14, 15, 16] it is possible to describe a state in quantum mechanics by means of positive marginal distribution function instead of wave function or density matrix (or instead of Wigner function or Husimi function) because the marginal distribution determines completely Wigner function (density matrix). The aim of the work is to introduce a tomography approach to determine the particle beam state in the framework of quantum-like TWM and to use positive marginal probability distributions in quantum-like domain of the particle-beam behavior.

2 BEAM TOMOGRAPHY IN QUANTUM-LIKE DOMAIN: A FOKKER-PLANCK-LIKE EQUATION FOR BEAM EVOLUTION

In this section, we obtain the evolution equation describing the particle beam in terms of a Fokker–Planck-type equation for positive probability distribution function. We also obtain this beam-evolution equation for potential of practical interest (quadrupole plus multipoles).

It is well known from quantum mechanics that the Wigner function [17] represents the nonnegative density operator $\widehat{\rho}$ [18, 19] in a particular representation. It is Hermitian, i.e. $\widehat{\rho}^\dagger = \widehat{\rho}$, and its trace is equal to unity: $\mathrm{Tr}\,\widehat{\rho} = 1$. For any representation, diagonal elements of the density operator are nonnegative, since they describe probability distribution function in a corresponding basis. In coordinate representation, we have $\langle x \mid \widehat{\rho} \mid x\rangle = P(x)$, where $P(x)$ is the position distribution function. The Wigner-like function ρ_w, satisfying Eq.(3), is related to the density matrix in coordinate representation by invertible transform (hereafter we take $\epsilon = 1$) [7]:

$$\rho_w\,(q,\,p) = \int \langle q + \frac{u}{2} \mid \widehat{\rho} \mid q - \frac{u}{2}\rangle \exp\left(-ipu\right)\,du \quad (4)$$

and

$$\langle x \mid \widehat{\rho} \mid x'\rangle = \frac{1}{2\,\pi}\int \rho_w\left(\frac{x+x'}{2},\,p\right) e^{ip(x-x')}\,dp\,. \quad (5)$$

On the basis of the Ref.[20], it is possible to prove that, for any Hermitian operator $\widehat{X}$, the Fourier transform of a characteristic function $\chi(k) \equiv \langle \exp\left(ik\widehat{X}\right)\rangle = \mathrm{Tr}\,\widehat{\rho}\exp\left(ik\widehat{X}\right)\rangle$ i.e.,

$$w(y) = \frac{1}{2\,\pi}\int \chi(k)\exp\left(-iky\right)\,dk$$

is the distribution function with classical features. In fact, by taking into account the positivity of the diagonal elements of the density operator one can see that $w(y) = \langle y \mid \widehat{\rho} \mid y\rangle \geq 0$ and $\int w(y)\,dy = 1$. By considering a specific operator $\widehat{X}$ of the form $\widehat{X} = \mu\hat{q} + \nu\hat{p}$, one can write that:

$$w\,(X,\,\mu,\,\nu) = \int \rho_w(q,\,p)\,e^{-ik(X-\mu q-\nu p)}\,\frac{dk\,dq\,dp}{(2\pi)^2}\,, \quad (6)$$

i.e., $w\,(X,\,\mu,\,\nu) \geq 0$, and $\int w\,(X,\,\mu,\,\nu)\;dX = 1$. In analogy with quantum optics, we call this function *quantum-like marginal distribution of the particle beam.* Note that we have used here the property of the Wigner distribution function

$$\mathrm{Tr}\,\widehat{\rho}\,e^{ik(\mu\hat{q}+\nu\hat{p})} = \int \rho_w\,(q,\,p)\,e^{ik(\mu q+\nu p)}\,\frac{dq\,dp}{2\,\pi}\,. \quad (7)$$

Formula (6) can be inverted

$$\rho_w(q,\,p) = \frac{1}{2\pi}\int w\,(X,\,\mu,\,\nu)\,e^{-i(\mu q+\nu p-X)}\,d\mu\,d\nu\,dX\,. \quad (8)$$

For beam dynamics, one can construct an equation in terms of marginal distribution following [13]. For the quantum-like Hamiltonian $H = \hat{p}^2/2 + U(x,z)\ ,\ \ \hat{p} = -i\,\partial/\partial x\,$, we obtain from (3), in view of (6), the following equation ($\epsilon = 1$):

$$\left\{\frac{\partial}{\partial z} - \mu\frac{\partial}{\partial\nu} - i\widehat{DU}\left(-\frac{1}{\partial/\partial X}\frac{\partial}{\partial\mu}\,,i\frac{\nu}{2}\frac{\partial}{\partial X},z\right)\right\} w = 0\,. \quad (9)$$

For the case of quadrupole-like potential (linear lens), i.e. $U(x,z) = k_1(z)x^2/2$ (k_1 being the quadrupole strength), we have

$$\frac{\partial w}{\partial z} - \mu\,\frac{\partial}{\partial\nu}\,w + k_1\,(z)\,\nu\,\frac{\partial}{\partial\mu}\,w = 0\,. \quad (10)$$

Gaussian solutions of (10) have the following form:

$$w\,(X,\mu,\nu,z) = \frac{1}{\sqrt{2\pi\sigma_X(z)}}\,\exp\left\{-\frac{(X-\bar{X})^2}{2\sigma_X(z)}\right\}, \quad (11)$$

in which $\bar{X} = \mu\langle q\rangle + \nu\langle p\rangle$.

In general, the evolution equation (9) for the marginal distribution of the particle beam can be cast in the form

$$\frac{\partial w}{\partial z} - \mu\frac{\partial w}{\partial\nu} +$$

$$+\,2\sum_{n=0}^{\infty}\frac{U^{2n+1}\left(\widetilde{q}\right)}{(2n+1)!}\left(\frac{\nu}{2}\,\frac{\partial}{\partial X}\right)^{2n+1}(-1)^{n+1}\,w = 0\,,$$

where $U^{2n+1}(\widetilde{q}) = \partial^{2n+1}U/\partial q^{2n+1}\,(q=\widetilde{q})$, with the operator $\widetilde{q}$ given by $\widetilde{q} = -\left(\partial/\partial X\right)^{-1}\partial/\partial\mu$. This evolution equation in quantum-like domain can also be presented in the form

$$\frac{\partial w}{\partial z} - \mu\,\frac{\partial w}{\partial \nu} - \frac{\partial U}{\partial q}(\widetilde{q})\,\nu\,\frac{\partial w}{\partial X} +$$

$$+\,2\sum_{n=1}^{\infty}\frac{U^{2n+1}(\widetilde{q})}{(2n+1)!}\left(\frac{\nu}{2}\,\frac{\partial}{\partial X}\right)^{2n+1}(-1)^{n+1}w \;=\; 0\,.$$

For potentials of the form

$$U(x,z) = \frac{1}{2!}k_1(z)x^2 + \frac{1}{4!}k_3(z)x^4 + \frac{1}{6!}k_5(z)x^6 \qquad (12)$$

(k_3, and k_5 being the multipole strengths), the corresponding quantum-like evolution equation for the marginal distribution of the particle beam has the form

$$\frac{\partial w}{\partial z}\;-\;\mu\frac{\partial w}{\partial \nu} + k_1(z)\nu\frac{\partial w}{\partial \mu} +$$

$$+\,\frac{k_3(z)}{6}\left[\nu\left(\frac{\partial}{\partial\mu}\right)^3\left(\frac{\partial}{\partial X}\right)^{-2} + \frac{\nu^3}{4}\,\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^2\right]\,w\,+$$

$$+\,\frac{k_5(z)}{120}\left[\nu\left(\frac{\partial}{\partial\mu}\right)^5\left(\frac{\partial}{\partial X}\right)^{-4}\right]\,w +$$

$$+\,\frac{k_5(z)}{120}\left[-\,\frac{5\,\nu^3}{6}\left(\frac{\partial}{\partial\mu}\right)^3 - \frac{\nu^5}{16}\,\frac{\partial}{\partial\mu}\left(\frac{\partial}{\partial X}\right)^4\right]\,w \;=\; 0\,.$$

3 CONCLUSIONS

In this paper we have described the particle beam transport in accelerators both in conventional classical context and in non-conventional quantum-like domain with TWM. We have shown that in quantum-like domain this function satisfies a Fokker–Planck-like equation for a positive marginal distribution function. The method of measuring the particle-beam state suggested here is fully similar to the one elaborated in quantum optics for optical and symplectic tomography. In the tomography formalism, the beam evolution equation for marginal distribution including multipole terms has been presented. Note that, when the quadrupole potential is considered without higher-order multipoles, the marginal distribution of squeezed coherent states can be obtained explicitly. According to the procedure used in this paper to determine an evolution equation for the quantum-like marginal distribution distribution, one can obtain a similar description also in classical domain. For example, the evolution equation for the potential given by (12) in classical domain would be:

$$\frac{\partial w}{\partial z} - \mu\frac{\partial w}{\partial \nu} + k_1(z)\nu\frac{\partial w}{\partial \mu} + \frac{k_3(z)}{6}\nu\left(\frac{\partial}{\partial\mu}\right)^3\left(\frac{\partial}{\partial X}\right)^{-2}\,w +$$

$$+\,\frac{k_5(z)}{120}\nu\left(\frac{\partial}{\partial\mu}\right)^5\left(\frac{\partial}{\partial X}\right)^{-4}\,w\;=0\,.$$

One sees that this equation can be obtained from the quantum-like one if the terms with high powers of the parameter ν are omitted. These terms are responsible for the quantum-like corrections to the beam behaviour.

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