# ON ENERGY EQUIPARTITION INDUCED BY SPACE CHARGE IN BUNCHED BEAMS

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*Abstract*

Space-charge interactions in high-intensity linear accelerator can lead to equipartitioning of energy between the longitudinal and the radial degrees of freedom. This can be a source of emittance growth and halo formation. Equipartitioning phenomena is analyzed. The excited coupling-resonances are described as a function of both beam and accelerator parameters.

## 1 INTRODUCTION

P. Lapostolle first discussed 30 years ago the question of emittance exchange between the radial and longitudinal degrees of freedom in high-intensity proton linacs [1]. After a straightforward analogy with heat exchanges, he immediately pointed out that ″this has no physical sense″. Actually, the fact that the mean time between collisions is much longer than the time spent by particles in a typical linac excludes a direct interpretation in terms of statistical mechanics. Nevertheless, M. Promé [2] did numerical simulations showing emittance exchange and an evolution towards an equipartitioning (EQP) of the radial (x) and longitudinal (y) mean kinetic energies:

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle \quad (1)$$

The fact that this equilibrium was reached more rapidly as the beam current increased has been interpreted as a direct effect of the nonlinear space charge forces. But, this could well be due to mismatch increasing with the beam current! R.A. Jameson [3] confirmed these results and pointed out that for a matched beam, eq.(1) leads to :

$$\varepsilon_x / \varepsilon_y = \sigma_y / \sigma_x = a / b \quad (2)$$

where $\varepsilon_x$ and $\varepsilon_y$ are the rms emittances, $\sigma_x$ and $\sigma_y$ are the phase advances with space charge, a and b the rms bunch radii in the transverse and longitudinal degrees of freedom respectively. Jameson also did a simulation showing that no emittance exchange occurs when a linac is designed using the EQP rule (eq. 2). However, this was unfortunately done for a very special case where the ratio of eqn. 2 equals to 1, i.e., a round beam (a = b) where the space charge forces cannot induce (x-y) coupling !

The ″physical sense″ of equipartition for practically collisionless beams has not been thoroughly analyzed since then. Equipartition, however, has sometimes been emphasized to the level of a basic rule to avoid not only emittance exchanges but also emittance growth and halo formation. Assertions aside, several important questions can be raised about the EQP rule (2) :

- It is based on molecular chaos or ergodic hypothesis at the microscopic level. Is it justified in our collisionless N-particle systems ? If yes, for which beam and accelerator parameters ?

- It refers to transverse and longitudinal rms values, as if the system could be reduced to a pair of coupled oscillators. What is the link with the classical EQP theory [4] which applies to the eigen modes of oscillation (if they are well separated) and not to individual oscillators ?

- We know that the space charge forces cannot induce any coupling for a round beam considered in Ref 3. How the space charge forces can then be responsible for EQP ?

- How is it that several high current linacs designed with apparent violation of the EQP rule show absolutely no sign of evolution towards an equipartitioned equilibrium?

- Its application can lead to strong tune depressions which is well known to have catastrophic consequences on the beam dynamics. What is preferable, an equipartition with severe tune depression or non-severe tune depression without equipartition ?

Answers to these questions are not straightforward and need careful work. In the following sections, we present some ideas and topics which appear relevant and could serve as the starting point for work in this direction. Section 2 discusses basic physics linked to equipartition while section 3 deals on the classical problem of modern physics, the FPU problem, which is the paradigm for EQP. In section 4, we present studies on the coupling resonances and new results.

## 2 EQUIPARTITION

The theorem of energy-equipartition [5] is easily understood for a thermodynamical system where each degree of freedom exchanges its excess energy through multiple collisions. The relaxation time to reach the "equipartitioned" equilibrium state is directly determined by the collision rate between the large-N molecules enclosed in a finite volume.

Equipartition in generic Hamiltonian systems is much more complicated and still a subject of intensive research in several fields of Physics. The energy-equipartition is generally based on the assumption that the system is ergodic. In this case, trajectories in the system must cover all parts of phase space, and be able to pass arbitrarily close to any point in the phase space infinite number of times. Thus, a system with invariant KAM curves is not ergodic since a part of the system is confined to a localized phase space region. Nevertheless, if the dynamics is chaotic in some subspaces of the phase space, the system can be considered as ergodic in these subspaces [6]. In this case, the system contains a mixture of quasiperiodic (KAM) and chaotic trajectories, and a mixture of stable and unstable periodic orbits (fixed points). It is this regime of weak chaos which is more difficult to analyze than the one of strong chaos for which the fast mixing property induces an ergodic behavior with energy equipartition.

For proton beam currents up to 100 mA or more, the tune depressions $\eta = \sigma/\sigma_o$ is usually greater than 0.2 and fortunately most of the particle trajectories are stable ($\sigma$ and $\sigma_o$ are the phase advances with and without space charge respectively). The ergodic hypothesis is verified only in some narrow chaotic regions of phase space (see [7] and references therein). The system of particles under such conditions is in a state of weak chaos with weak mixing properties.

## 3 THE FPU PROBLEM

For a system of N- coupled oscillators, the equipartition theorem states that every frequency would on the average, have the same energy, if averaged over a sufficiently long time. The determination of this relaxation time as a function of the system parameters i.e., number of oscillators, mean-energy per oscillator, frequency of the initial excitation, has been the subject of studies since the very beginnings of statistical mechanics.

The Fermi-Pasta-Ulam problem addresses the question of energy-equipartition in a chain of coupled oscillators with nonlinear coupling. To understand the ‘asymptotic’ or large-time scale behavior of a nonlinear system, they did numerical experiment with a chain of 64 particles [8]. The result was contrary to the expectation: The energy given to one of the normal modes was exchanged in a complicated but recurrent way among all other modes but no equipartition of energy was observed. This famous numerical experiment done using one the first computers available after the second world war was then called the "FPU problem" since the expected "thermalization" did not occur.

Subsequent extensive numerical simulations with the advent of powerful computers confirmed the existence of an energy threshold above which the motion becomes fully chaotic and quick mixing takes place i.e., in the physical sense, the large N-system like the FPU lattice can be considered ergodic (see [9], [6] and references therein). Above this energy threshold known as strong stochastic threshold (SST), the property that sharply changes is the mixing rate which in turn is directly related to the strength of chaoticity (weak or strong) of the system. An unambiguous definition of the SST is given by the crossover value $\varepsilon_c$ ($\varepsilon$ is the energy per degree of freedom) of the largest Lyapunov exponent [6]. The SST provides interesting insight into the energy dependence of the relaxation time needed to reach the equilibrium (equipartition) from an initial state in a large N-body nonlinear Hamiltonian system.

Fig.1 [9] shows, as an example, the $\varepsilon$-dependence of the relaxation time $\tau_R$ and the maximum Lyapunov exponent $\lambda_1$ (full circles) for a FPU model with N=128 where four lowest modes are initially excited. Dashed lines represent power laws $\varepsilon^2$ and $\varepsilon^{2/3}$. Open circles and squares represent relaxation times to equipartition of energy among normal modes of the energy initially given to the four lowest modes of the chain. The crossover point of $\lambda_1$ clearly defines the strong stochastic threshold. At high energy (the domain of strong chaos) $\tau_R$ is almost independent of the energy. At low energy (weak chaos regime) $\tau_R$ strongly increases as the energy decreases. It should be noted here that SST is an intrinsic character independent of the initial condition of the system.

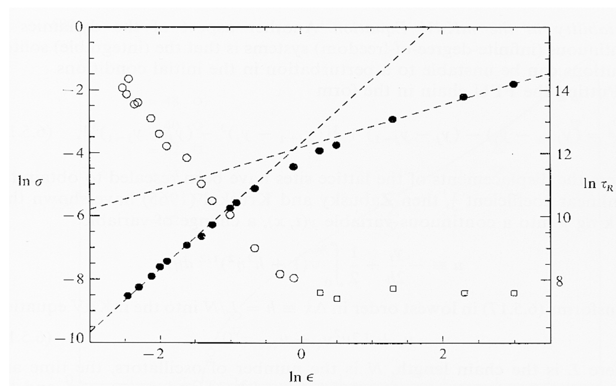


Fig. 1 : Largest Lyapunov exponent and relaxation time in the FPU problem ([9], see also the text above and [6])

A system of N-bunched particles constrained by external and space-charge forces can be viewed as a system of N-coupled oscillators. It seems that the techniques used for the FPU problem could be fruitfully used to study the relaxation time towards an equipartitioned equilibrium in such systems. As already mentioned, proton beams with currents up to 100 mA or more usually lead to a dynamics with weak chaos and weak mixing. Thus, the working point is probably below the Strong Stochastic Threshold! This remains to be verified. An analysis with different parameters to find the sensitivity of the SST to parameters will also be very useful.

## 4 COUPLING RESONANCES

The emittance transfer mechanism induced by coupling resonances is well known in circular accelerators. I. Hofmann has studied this source of ″collective instabilities″ induced by space charge. He has drawn charts and identified the instability thresholds (SST ?) for the main modes by solving the Vlasov's equation [10] [12]. Hofmann’s multiparticle simulations as well as others done by Jameson [11] clearly demonstrated the utility of these charts with respect to phase space dilution. Starting from the equations used in [12], we will show that unstable areas correspond to locations in Hofmann's chart where the coupling resonances can be excited.

The dynamics of a matched beam with radial symmetry and under smooth approximation can be defined by three beam parameters ($\varepsilon_x$, $\varepsilon_y$ and intensity) and two parameters related to the external focusing forces ($\sigma_{ox}$ and $\sigma_{oy}$). It can be easily shown that after a normalization, the beam dynamics can be studied without any loss of generality using only 3 of the 5 following parameters :

$$\eta_y = \sigma_y/\sigma_{oy} \quad \alpha = \sigma_x/\sigma_y \quad \varepsilon_x / \varepsilon_y \quad \eta_x = \sigma_x/\sigma_{ox} \quad a / b$$

This is illustrated in Fig.2 which shows the curves for $\eta_x$ = constant as a function of $\eta_y$ and $\alpha$ with $\varepsilon_x/\varepsilon_y = 5$.

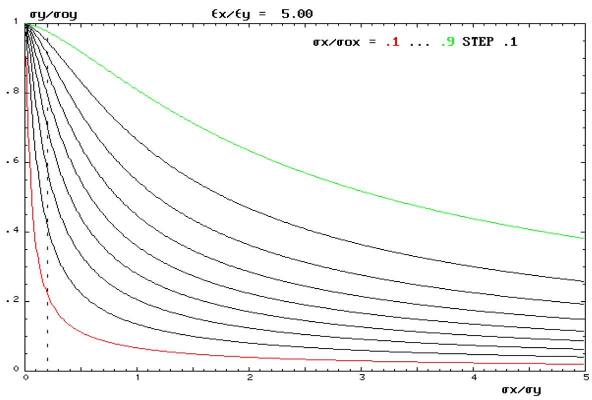


Fig 2: Curves of $\eta_x$ = constant vs $\alpha$ and $\eta_y$ with $\varepsilon_x/\varepsilon_y = 5$.

For a given point ($\alpha$, $\eta_y$) and $\varepsilon_x/\varepsilon_y$, the 4 tunes $\sigma_x$ (normalized to 1), $\sigma_{ox}$ , $\sigma_y$ and $\sigma_{oy}$ are then known. The area occupied by the beam with space charge in the tune diagram (Fig.3) for each ($\alpha$, $\eta_y$) is then defined.

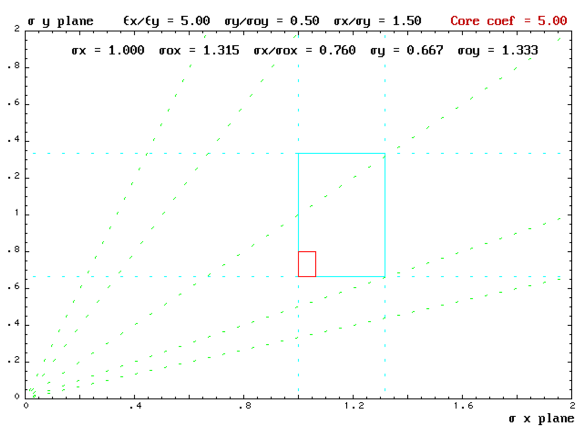


Fig. 3 : Tune diagram for $\alpha = 1.5$, $\eta_y = 0.5$, $\varepsilon_x/\varepsilon_y = 5$. The small red area represents the beam core.

It is easy to determine if a coupling resonance is present for the beam area from Fig.4 which is a chart for the parameters ($\alpha$, $\eta_y$) with $\varepsilon_x/\varepsilon_y = 5$. This chart is very similar to the one calculated by I. Hofmann.

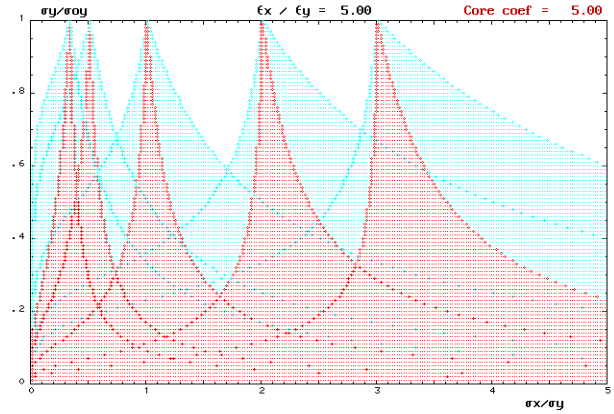


Fig. 4 : Chart giving the areas where the coupling resonances 1/3, 1/2, 1/1, 2/1 and 3/1 can be excited

## 5 COMMENTS

It must be noted that the underlying physics of analysis in terms of coupling resonances is close to the one used in the "FPU problem". The concepts of resonance, resonance overlap leading to weak or strong chaos depending on the excitation energy are good candidates to explain why and when EQP may occur. It should also be pointed out that these concepts are far from those leading to the EQP rule (Eq.2), although the two approaches get mixed in some publications. Furthermore, a working point which satisfy the EQP rule can very well be below the stability threshold in the Hofmann's chart !

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