# RF PARAMETERS FOR RECTANGULAR ACCELERATING STRUCTURES

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*Abstract*

Fabricational techniques for very high frequency structures require planar geometries. Typical devices are ladder or muffin-tin structures which are both essentially rectangular. Although a specific geometry has to be calculated numerically, it is often convenient to use approximations or to have a very fast code for an interactive approach. We present a field-matching code for rectangular periodic structures which computes the basic RF parameters, such as frequency, R upon Q (longitudinal and transverse), Q-value, attenuation, Brillouin-diagram and field pattern in a fast way. We also give analytical dependencies of the longitudinal and transverse voltage gain and some approximations of RF parameters.

## 1 INTRODUCTION

Recently, planar structures have been considered for particle acceleration at very high frequencies, around 100 GHz [1], [2].It was proposed to fabricate the structures either by means of deep x–ray lithography (LIGA) or by wire electro–discharge–machining (EDM). In both cases the structures are basically muffin–tin structures with rectangular cross–sections. Their RF–parameters have to be calculated numerically with codes like GdfidL [5], for instance. The standard structure can also be calculated with a mode–matching technique [3], [4]. But often a very fast code or only approximate solutions are convenient. Therefore we have developed a mode–matching code for a simplified geometry which is purely two–dimensional, Fig 1. The code allows for the 2.5–dimensional computation of fields with plots and the main RF–parameters like frequency, Q–value and longitudinal and transverse shunt impedance, all for a given phase advance per cell. Additionally, approximate analytical expressions have been derived for the structure frequency and the longitudinal and transverse voltage gains of a probing particle.

## 2 MODE–MATCHING SOLUTION FOR A 2–D RECTANGULAR STRUCTURE

Let us take a purely rectangular structure, Fig.1, which is for many purposes a good approximation of a muffin–tin structure.

The mode–matching solution for such a structure is straight forward and will be given here only in short. We separate the volume into two regions, the beam region 1 and the cavity region 2. In the beam region we use space–harmonics

$$A^{(1)} = \sum_{n=-\infty}^{\infty} A_n \left\{ \begin{array}{c} \cosh \\ \sinh \end{array} \right\} (\tau_n y) \cos k_x x \; \mathrm{e}^{\,-\mathrm{j}\beta_n z} \qquad (1)$$

with $k_0^2 = k_x^2 - \tau_n^2 + \beta_n^2$, $\beta_n = \beta_0 + n\frac{2\pi}{L}$, $k_x = M\frac{\pi}{w}$. $M$ even relates to magnetic walls and $M$ odd to electric walls at $x = \pm w/2$. The standing–waves in the cavity region are

$$A^{(2)} = \sum_{n=0}^{\infty} B_n \cosh\left(\lambda_n [y-b]\right) \cos k_x x \cos \frac{n\pi z}{g} \qquad (2)$$

with $k_0^2 = k_x^2 - \lambda_n^2 + \left(\frac{n\pi}{g}\right)^2$. Since the geometry is only 2–dimensional, it is sufficient to consider modes with an $H_x$–component only and no $E_x$–component. Then the corresponding $E$– and $H$–components can be calculated from equs. (1) and (2)

$$\mathbf{E} = \nabla \times (A\,\mathbf{e}_x) = \frac{\partial A}{\partial z}\mathbf{e}_y - \frac{\partial A}{\partial y}\mathbf{e}_z \qquad (3)$$

$$-\mathrm{j}\omega\mu\mathbf{H} = -\left[k_x^2 - k_0^2\right] A\,\mathbf{e}_x + \frac{\partial^2 A}{\partial x \partial y}\mathbf{e}_y + \frac{\partial^2 A}{\partial x \partial z}\mathbf{e}_z \; .$$

At the interface between 1 and 2, at $y = \pm a$, the tangential field components have to be matched. That means, we determine the Fourier coefficients $A_n$ such that $E_z^{(1)}$ equals $E_z^{(2)}$ in the cavity gaps, $nL \le z \le nL + g$, $-\infty < n < \infty$, and is zero on the irises, $-t + nL \le z \le nL$. For the match of the $H_x$–component we determine the coefficients $B_n$ such that $H_x^{(2)}$ equals $H_x^{(1)}$ in the cavity gap only. Substituting one set of equations into the other, leads to an infinitely large set of homogeneous linear equations. For $n$ large enough, the system converges and can be approximated by a finite system. Its eigenvalues $\tilde{\beta}_{0i}$ and eigenvectors $\tilde{\mathbf{A}}_i$ are approximations of the propagation constant

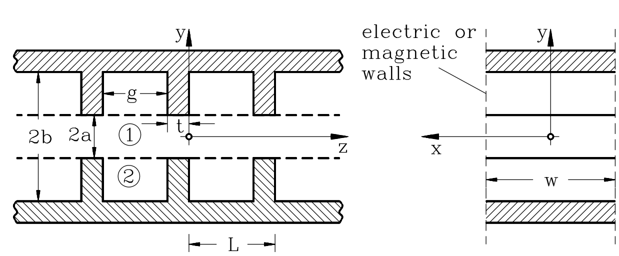


Figure 1: Longitudinal and transverse cross–sections of a rectangular structure.

$\beta_{0i}$ and the expansion coefficients $A_{ni}$, respectively, of mode $i$. Thus the fields are fully determined and the basic RF–parameters, like shunt impedance, $Q$–value, attenuation constant, group velocity etc., are easily calculated. We have written a simple and fast computer code, called PST, which calculates the RF–parameters and plots the contour lines of constant $H_x$ (which equal the lines of force in case of $M = 0$) for a given phase advance per cell and a given geometry. As an example the Figures 2 and 3 show the dispersion relations and the field plots of the first accelerating and deflecting modes of a 120 GHz, $2\pi/3$–mode structure.

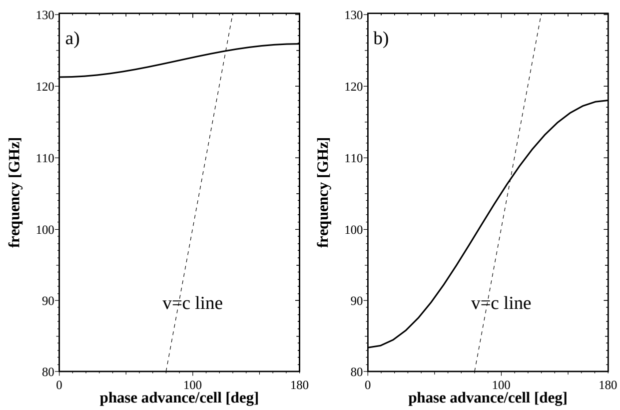


Figure 2: Dispersion diagram of the first a) accelerating and b) deflecting mode of a 120 GHz, $2\pi/3$–mode structure.

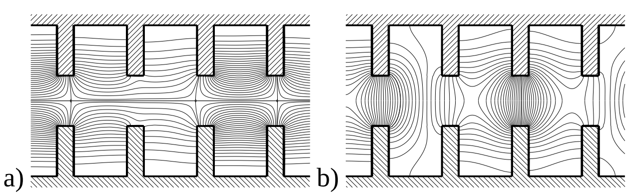


Figure 3: Lines $H_x =$ const. of the first a) accelerating and b) deflecting mode of a 120 GHz, $2\pi/3$–mode structure ($v_{ph} = c_0$).

## 3 ANALYTICAL EXPRESSIONS FOR LONGITUDINAL AND TRANSVERSE VOLTAGE GAINS

If the fields are dependent on the transverse coordinate $x$ or if the probing particle has a velocity smaller than the velocity of light, the voltage gain experienced by the particle depends on the transverse position of the trajectory. A good approximation of this dependence for a typical accelerating mode can be derived by assuming a single mode in the cavity region

$$A^{(2)} = B_0 \cosh \lambda_0 (y - b)\ \cos k_x x\ ,\ \lambda_0^2 = k_x^2 - k_0^2\,. \quad (4)$$

In average, the particle interacts only with the synchronous space–harmonic, which we assume to be the fundamental

$$A^{(1)} = A_0 \sinh \tau_0 y\ \cos k_x x\ \mathrm{e}^{-\mathrm{j} k_0 z/\beta} \quad (5)$$

$\tau_0^2 = k_x^2 + \left(\frac{k_0}{\beta\gamma}\right)^2$. At the interface $y = a$, we require $E_z^{(1)}$ to be zero on the irises and to be continuous with $E_z^{(2)}$ over the gap region. This yields

$$A_0 \tau_0 L\ \cosh \tau_0 a = B_0 \lambda_0 g \sinh \lambda_0 (a - b)\ \mathrm{e}^{\mathrm{j}\alpha} \frac{\sin\alpha}{\alpha} \quad (6)$$

$\alpha = \frac{k_0 g}{2\beta}$. From equ.(5) together with (6) follows the longitudinal voltage gain per structure length as

$$V_z' = \frac{1}{L} \int_0^L E_z^{(1)}\ \mathrm{e}^{\mathrm{j} k_0 z/\beta}\, \mathrm{d}z = \ \mathrm{e}^{\mathrm{j}\alpha} B_0 \lambda_0 \frac{g}{L} \cos k_x x\ RT \quad (7)$$

with $R = \sinh \lambda_0 (b - a) \frac{\cosh \tau_0 y}{\cosh \tau_0 a}\ ,\ T = \frac{\sin\alpha}{\alpha}$.

Several factors enter into equ.(7). Apart from the constant $\mathrm{e}^{\mathrm{j}\alpha} B_0 \lambda_0$, there is the ratio of gap to period length, the assumed $x$–dependence, a transit time factor $T$ which takes the finite traveling time across the gap into account, and finally a factor $R$ which is due to the exponential decay of the field across the aperture.

In a similar way we obtain the transverse voltage gains as

$$\begin{aligned} V_x' &= -\frac{1}{L} \int_0^L v\mu H_y\ \mathrm{e}^{\mathrm{j} k_0 z/\beta}\, \mathrm{d}z = \mathrm{j}\, \frac{\beta}{k_0} \frac{\partial V_z'}{\partial x} \\ V_y' &= \frac{1}{L} \int_0^L (E_y + v\mu H_x)\ \mathrm{e}^{\mathrm{j} k_0 z/\beta}\, \mathrm{d}z = \mathrm{j}\, \frac{\beta}{k_0} \frac{\partial V_z'}{\partial y} \quad (8) \end{aligned}$$

## 4 APPROXIMATION FOR STRUCTURE FREQUENCY AND $R$ UPON $Q$

Often it is convenient to get a fast estimate of the structure frequency or of the geometrical parameters for a given frequency.

A 0–order approximation takes into account only the fundamental space–harmonic in the beam region, equ.(1), and the $z$–independent mode in the cavity region, equ.(2). Then we have different options to set up an eigenvalue equation. The first and commonly used way is to equate the transverse wave impedances at the interface $y = a$

$$E_z^{(1)} / H_x^{(1)} = E_z^{(2)} / H_x^{(2)}\,. \quad (9)$$

Using equs.(1) and (2) in (3) and substituting into (9) yields finally an equation equal to the eigenvalue equation of a dielectric slab

$$pu = \cot u \quad (10)$$

with

$$p = \frac{1}{b/a - 1} \frac{\tanh \tau_0 a}{\tau_0 a}\ ,\ \tau_0 a = \sqrt{(k_x a)^2 + \left(\frac{k_0 a}{\beta\gamma}\right)^2}$$

$$u = \left(1 - \frac{b}{a}\right)\sqrt{(k_0 a)^2 - (k_x a)^2}\,.$$

Equ.(10) can be solved numerically but also analytically after evaluating the cot–function. As a result we obtain the normalized frequency

$$k_0 b = \sqrt{\left(\frac{u_0}{1 - a/b}\right)^2 + (k_x b)^2} \qquad (11)$$

where

$$u_0 \approx \frac{\pi}{2} - \left[\frac{3\pi}{4}p + D\right]^{1/3} - \left[\frac{3\pi}{4}p - D\right]^{1/3}$$

$$D^2 = (1+p)^3 + \left(\frac{3\pi}{4}p\right)^2 .$$

Another possibility to derive an eigenvalue equation is to use the first two equations of the mode matching procedure mentioned in chapter 2. The first equation is equ.(6) and the second follows from matching $H_x^{(2)}$ to $H_x^{(1)}$ over the gap length

$$A_0 \sinh \tau_0 a \, \mathrm{e}^{-\mathrm{j}\alpha} \frac{\sin\alpha}{\alpha} = B_0 \cosh \lambda_0 (a - b)\,. \qquad (12)$$

Eliminating the constants in the equs.(6) and (12) results in a similar equation to equ.(10) but with an additional factor

$$\frac{g}{L}\left(\frac{\sin\alpha}{\alpha}\right)^2 pu = \cot u \ , \ \alpha = \frac{k_0 g}{2\beta}\,. \qquad (13)$$

Both 0–order approximations are shown in Fig.4 together with the exact solution obtained by the code PST. Obviously, the 0–order approximations give estimates with

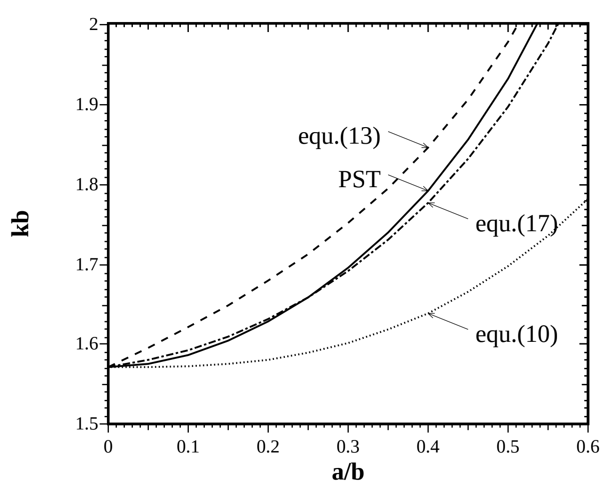


Figure 4: Normalized frequency versus $a/b$ for the accelerating mode with $M = 0$ and $v_{ph} = c_0$.

about 5% error in case of equ.(13) and even more in case of equ.(10). For better approximations we go one step further and add the minus–one space–harmonic in the beam region

$$\begin{aligned} A^{(1)} &= A_{-1} \sinh \tau_{-1} y \, \cos k_x x \, \mathrm{e}^{-\mathrm{j}\beta_{-1} z} \\ &+ A_0 \sinh \tau_0 y \, \cos k_x x \, \mathrm{e}^{-\mathrm{j}k_0 z/\beta} \end{aligned} \qquad (14)$$

$\beta_{-1} = \frac{k_0}{\beta} - \frac{2\pi}{L}$. To match the tangential electric field at $y = a$ we proceed like in chapter 3. Then, the Fourier coefficient $A_0$ is given in equ.(6) and for $A_{-1}$ we obtain

$$\begin{aligned} & A_{-1}\tau_{-1} L \, \cosh \tau_{-1} a = \\ = \;& B_0 \lambda_0 g \, \sinh \lambda_0 (a - b) \, \mathrm{e}^{\mathrm{j}\alpha_{-1}} \frac{\sin\alpha_{-1}}{\alpha_{-1}} \end{aligned} \qquad (15)$$

$\alpha_{-1} = \frac{k_0 g}{2\beta} - \frac{\pi g}{L}$. The matching of $H_x^{(2)}$ to $H_x^{(1)}$ at $y = a$ yields now

$$\begin{aligned} & A_{-1} \sinh \tau_{-1} a \, \mathrm{e}^{-\mathrm{j}\alpha_{-1}} \frac{\sin\alpha_{-1}}{\alpha_{-1}} + A_0 \sinh \tau_0 a \, \mathrm{e}^{-\mathrm{j}\alpha} \frac{\sin\alpha}{\alpha} \\ & = B_0 \cosh \lambda_0 (a - b)\,. \end{aligned} \qquad (16)$$

We eliminate the constants $A_{-1}$, $A_0$, $B_0$ from the equs.(6), (15), (16) and obtain the 1st–order eigenvalue equation

$$\left[p_{-1} T_{-1}^2 + p T^2\right] \frac{g}{L} u = \cot u \qquad (17)$$

were $u$ and $p$ are given in equ.(10) and $T$ is given in equ.(7). $p_{-1}$ and $T_{-1}$ are equal to $p$ and $T$ but with $\beta_0$ replaced by $\beta_{-1}$, i.e.

$$(\tau_{-1} a)^2 = (k_0 a)^2 - (k_x a)^2 + (\beta_{-1} a)^2 \ , \ \alpha_{-1} = \beta_{-1} g/2\,.$$

As can be seen from Fig.4, the eigenvalues are now precise within 2% or less.

The above approximations can also be used to calculate other RF–parameters like $Q$–value and $R/Q$. While the 0–order approximation yields values which are not very useful, the 1st–order approach is already quite effective. However, the formulas are lengthly and will not be given here for space reasons. Rather we show results for a 90 and 120 GHz structure in Table 1.

| | 0–order | 1st–order | PST |
|---|---|---|---|
| $f/\mathrm{GHz}$ | 90.7 (120.9) | 93.5 (123.3) | 94.6 (124.8) |
| $\frac{R'/Q}{\mathrm{k}\Omega/\mathrm{m}}$ | – | 77.7 (133) | 76.1 (137) |
| $Q$ | – | 2670 (2280) | 2380 (2040) |

Table 1: Results for $2\pi/3$–mode structures
a=0.525 (0.3), b=1.27 (0.9), w=2.367 (1.8),
g=0.843 (0.633), t=0.25 (0.2) (all in mm).

## 5 REFERENCES


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