QUANTUM THEORY OF HIGH-GAIN FREE-ELECTRON LASERS *

R. Bonifacio, INFN-MI, Milano, Italy and Universidade Federal Alagoas, Maceiò, Brazil
N. Piovella[#], INFN-MI, Milan and Università degli Studi di Milano, Milan, Italy

Abstract

We formulate a quantum linear theory of the N-particle free-electron laser Hamiltonian model, quantizing both the radiation field and the electron motion, in the steady state regime. Quantum effects such as frequency shift, line narrowing, quantum limitation for bunching and energy spread and minimum uncertainty states are described. Using a second quantization formalism we demonstrate quantum entanglement between the recoiling electrons and the radiation field.

INTRODUCTION

Previous treatments [1-3] do not describe correctly the quantum FEL exponential regime because they define particles collective operators without the necessary symmetrization. In this paper we quantize both the electron motion and the radiation field in the steady state linear regime. Propagation and non linear effects quantizing only the electron motion have been described elsewhere [4]. The quantum behavior is ruled by a "quantum FEL parameter", $\bar{\rho}$ [1-3], which represents the ratio between the classical momentum spread and the one-photon recoil momentum. The classical limit is recovered only when this parameter is much larger than one. On the contrary, when $\bar{\rho} \le 1$, one has a shift of the resonant frequency and a narrowing of the gain bandwidth. We show the existence of a general uncertainty principle relating momentum spread and bunching, which implies that the maximum bunching is limited by energy spread. We define a minimum uncertainty state, which, for small fluctuations, reduces to a gaussian packet. Finally, the multi-particle approach is compared with a second-quantized approach in which the particles are described in terms of momentum states occupation operators. Photon statistics and *quantum entanglement* in an FEL, starting from vacuum fluctuations, are presented.

HAMILTONIAN MODEL

We start from the FEL Hamiltonian for N electrons interacting with a single mode of radiation [1-3]:

$$H = \sum_{j=1}^{N}\left[\frac{p_j^2}{2\bar{\rho}} + ig\left(a^+e^{-i\theta_j} - h.c.\right)\right] - \frac{\delta}{N}a^+a \qquad (1)$$

where $\theta_j = (k+k_w)z_j - ck - \delta\bar{z}$ and $p_j = mc(\gamma_j - \gamma_o)/\hbar(k+k_w)$ are position and momentum operators of the j-th electron, with $\left[\theta_i, p_j\right] = i\delta_{ij}$, a is the annihilation operator of the radiation field, with $\left[a, a^+\right] = 1$, $g = \sqrt{\bar{\rho}/N}$, $\bar{\rho} = \rho(mc\gamma_0/\hbar k)$ and ρ are, respectively, the quantum [1,3] and classical [5] FEL parameters, $\bar{z} = z/L_g$, $L_g = \lambda_w/4\pi\rho$ is the gain length and $\delta = (\gamma_r - \gamma_0)/(\rho\gamma_0)$ is the detuning. We observe that the dynamics depends on the single parameter $\bar{\rho}$. From Hamiltonian (1) we derive the following Heisenberg evolution equations:

$$\frac{d\theta_j}{d\bar{z}} = \frac{p_j}{\bar{\rho}};\ \frac{dp_j}{d\bar{z}} = -g\left(ae^{i\theta_j} + h.c.\right);\frac{da}{d\bar{z}} = g\sum_{j=1}^{N}e^{-i\theta_j} + i\delta a \qquad (2)$$

A constant of motion, which represents the total momentum in dimensionless units, is given by

$$\sum_{j=1}^{N} p_j + a^+a = const., \qquad (3)$$

Let us introduce the following electron collective operators

$$B = \frac{1}{\sqrt{N}}\sum_j e^{-i\theta_j}$$
$$P = \frac{1}{\sqrt{N}}\sum_j\left(\frac{p_je^{-i\theta_j} + e^{-i\theta_j}p_j}{2}\right) \qquad (4)$$

where B is the bunching and P is the symmetrized momentum bunching. This symmetrization is fundamental whenever one is dealing with non commuting operators, i.e., $\left[e^{-i\theta_j}, p_i\right] = \delta_{ji}e^{-i\theta_j}$.

We consider a, p_j and $\sum_j e^{-i\theta_j}$ as fluctuation operators, i.e. the initial states for the electrons and the field such that $\langle a\rangle_0 = \langle p_j\rangle_0 = \sum_j\langle e^{-i\theta_j}\rangle_0 = 0$.

Writing the Heisenberg equation of motion and neglecting the high-order quantities $\frac{1}{\sqrt{N}}\sum_j\left(p_je^{-i\theta_j}p_j\right)$ and $a^+\frac{1}{N}\sum_j e^{-2i\theta_j}$, we obtain the following equations for the linear regime:

$$\frac{dB}{d\bar{z}} = -\frac{i}{\bar{\rho}}P;\ \frac{dP}{d\bar{z}} = -\sqrt{\bar{\rho}}a - \frac{i}{4\bar{\rho}}B;\ \frac{da}{d\bar{z}} = \sqrt{\bar{\rho}}B + i\delta a \qquad (5)$$

*Work supported by INFN, Italy
[#]nicola.piovella@mi.infn.it

The quantum correction to the classical description [5] is given by the term $-iB/4\bar{\rho}$ in the equation for P. Looking for solutions of the linear system (5) of the form $B(\bar{z})=\exp(i\lambda\bar{z})B_0$, we obtain the cubic characteristic equation

$$(\lambda-\delta)(\lambda^2-1/4\bar{\rho}^2)+1=0 \qquad (6)$$

Notice that this dispersion relation coincides with that of a classical FEL with an initial energy spread with a square distribution and width $1/\bar{\rho}$ i.e., this extra term represents the *intrinsic quantum momentum spread* which, in dimensional units, becomes $\hbar k$. In [2,3] the linear approximation has been carried out without properly symmetrizing the momentum bunching operator, defined in (4). In fact, in [2] the momentum bunching is defined as $P_2=(1/\sqrt{N})\sum_j e^{-i\theta_j}p_j$. In the linear approximation the authors neglect the high-order term $\sum_j e^{-i\theta_j}p_j^2$, leading to the equation $dP_2/d\bar{z}=-a$ and to the classical cubic [5] equation $\lambda^2(\lambda-\delta)+1=0$, which can be obtained from Eq. (6) in the limit $\bar{\rho}>>1$. In [3] the authors define the momentum bunching as $P_3=(1/\sqrt{N})\sum_j p_j e^{-i\theta_j}$ without symmetrizing. Neglecting the high-order term $\sum_j p_j^2 e^{-i\theta_j}$, they obtain the following cubic equation (see Eq.(27) in [3]):

$$\lambda^3-(\delta+q)\lambda^2+(\delta q+q^2/4)\lambda+1-\delta q^2/4=0 \qquad (7)$$

where $q=1/\bar{\rho}$. However, defining $\lambda'=\lambda-q/2$ and $\delta'=\delta-q/2$, Eq.(7) becomes formally identical to the usual classical cubic [4] $\lambda'^2(\lambda'-\delta')+1=0$, just redefining the detuning. As a consequence, the analysis of the quantum corrections discussed in [3], in which the resonance is assumed for $\delta=0$, instead of $\delta=q/2=1/2\bar{\rho}$, is not correct. As a matter of fact the cubic equation which describes correctly the quantum behavior is not given by Eq.(7), but by Eq.(6), which has been obtained using the correct symmetrization of the collective operator, as given by Eq.(4). The features of the solution of the cubic equation (6) is shown in Fig.1.

When $\bar{\rho}\le 1$ (Fig. 1b-f), the instability rate decreases and the peak of $\mathrm{Im}\lambda$, i.e., the resonance, occurs around $\delta=1/2\bar{\rho}$ with peak value $\mathrm{Im}\lambda=\sqrt{\bar{\rho}}$ and full width on δ equal to $4\sqrt{\bar{\rho}}$. Note that the field and the bunching grow exponentially as $\exp(\sqrt{\bar{\rho}}\bar{z})=\exp(z/L_g')$, where $L_g'=L_g/\sqrt{\bar{\rho}}=\lambda_w/4\pi\rho\sqrt{\bar{\rho}}$ is the quantum gain length.

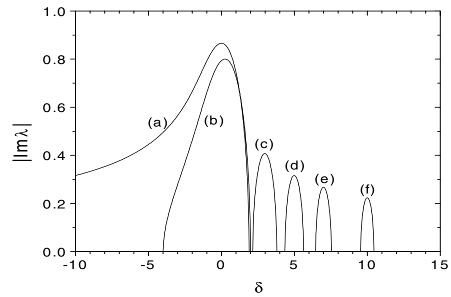

Figure 1: Imaginary part of the complex root of the cubic equation (6) vs. δ for $1/2\bar{\rho}$ equal to 0 (a), 0.5 (b), 3 (c), 5 (d), 7 (e) and 10 (f).

Hence, in the quantum regime $\bar{\rho}<<1$ the resonance frequency shifts to positive values as $1/2\bar{\rho}$, the gain length increases and the gain bandwidth narrows as the square root of the quantum FEL parameter $\bar{\rho}$. On the contrary, if one uses the cubic (7) of [3], one would obtain Fig.1a with the correct shift of the resonance to the right but all the other properties of the quantum solution are missing.

UNCERTAINTY RELATIONS

We now derive, from first principles, very general limitations for bunching and energy spread. The phase operator θ, defined in the (0,2π) space, and the canonically conjugate momentum $p=-i\partial/\partial\theta$, satisfy the commutation rule $[\theta,p]=i$. These two variables can be interpreted also as azimuthal angle about the z axis and z-component of the orbital momentum $L_z=p$, so that the momentum p (in units of $\hbar k$) has discrete eigenvalues $n=-\infty,..,\infty$ and normalized eigenfunctions $(1/\sqrt{2\pi})\exp(in\theta)$. As it is well known [6], assuming these discrete eigenstates, one *cannot* conclude that the commutation rule implies the uncertainty relation $\Delta\theta\Delta p\ge 1/2$. However, other uncertainty relations can be obtained using the periodic operators $\cos\theta$ and $\sin\theta$, with commutation rules with p: $[\sin\theta,p]=i\cos\theta$; $[\cos\theta,p]=-i\sin\theta$. Therefore, from the general uncertainty relations, one can deduce the following inequalities [6]:

$$\Delta p\Delta\sin\theta\ge(1/2)\langle\cos\theta\rangle$$
$$\Delta p\Delta\cos\theta\ge(1/2)\langle\sin\theta\rangle \qquad (8)$$

which can be combined in the single symmetrical relation:

$$(\Delta p)^2\left[(\Delta\cos\theta)^2+(\Delta\sin\theta)^2\right]\ge\frac{1}{4}\left[\langle\cos\theta\rangle^2+\langle\sin\theta\rangle^2\right] \qquad (9)$$

Defining the bunching $b = \langle e^{-i\theta} \rangle = \langle \cos\theta \rangle - i \langle \sin\theta \rangle$, Eq. (8) provides the following uncertainty relation between the momentum spread Δp (in units of k) and the bunching:

$$(\Delta p) \geq |b| / 2\sqrt{(1-|b|^2)} \quad (10)$$

which can be written also as:

$$|b| \leq \Delta p / \sqrt{\Delta p^2 + 1/4} \ . \quad (11)$$

This relation, derived from first principles, has an important physical consequence: it states that it is not possible to have large bunching without momentum spread, i.e. if |b| tends to unity, then the momentum spread becomes infinite. Conversely, if the momentum spread tends to zero, then the bunching is also zero. The first statement is with some respect intuitive, whereas the second one is quite surprising and it is a consequence of the Heisenberg uncertainty principle of Quantum Mechanics. Relation (11) set an upper limit to the maximum bunching obtainable in FELs, and states that |b| can be near unity only when $\Delta p >> 1/2$, i.e. when the momentum spread is much larger than ħk. In the case in which one can assume $\Delta\theta << 1$, one has $|b|^2 \approx 1 - (\Delta\theta)^2$ and the relation (10) reduces to the usual Heisenberg uncertainty principle $\Delta\theta\Delta p \geq (1/2)$.

It has been shown [7,8] that for $\bar{\rho} \leq 1$ the electrons, initially in the momentum state $p = 0$, can populate only the lower momentum state $p = -1$, recoiling backward by $\hbar k$ when a photon is emitted. In this quantum regime the Hilbert space is spanned by only two eigenstates of the discrete momentum, separated by ħk. Calling P_1 and P_2 the probability to be in the state with $n = 0$ or $n = -1$ (with $P_1+P_2=1$) it is easy to show that the momentum spread in units of ħk is $(\Delta p)^2 = P_1(1-P_1)$. Hence, the maximum spread occurs for $P_1=1/2= P_2$ and $\Delta p = 0.5$, and, using Eq. (11), it results that the maximum bunching must be limited by $1/\sqrt{2} \approx 0.71$. However, in the two-state approximation the bunching is [4] $|b|^2 = |c_0 c_{-1}^*|^2 = P_1 P_2 = \Delta p^2$ (because $P_1 = |c_0|^2$ and $P_2 = |c_{-1}|^2$), so that the maximum bunching in the two-state approximation is also 0.5, in agreement with the limitation given by Eq. (11).

We now introduce a minimum uncertainty state. It has been demonstrated [6] that there is no state that allows the symmetrical uncertainty relation Eq.(9), (11) to reach its minimum value. However, there exist states that minimize one of the two uncertainty relations (8). These minimum uncertainty states are solutions of the equation $(\partial/\partial\theta + \gamma\sin\theta)\psi_{min}(\theta) = i\lambda\psi_{min}(\theta)$ [6],[9]. The solution of ψ_{min} is

$$\psi_{min}(\theta) = G e^{\gamma\cos\theta + i\lambda\theta} \propto e^{-2\gamma\sin^2(\theta/2) + i\lambda\theta} \quad (12)$$

Because ψ must be single valued, $\lambda = m = \langle p \rangle$, $\langle \sin\theta \rangle = 0$ and G is the normalization constant, given by $G^{-2} = \int_0^{2\pi} d\theta e^{2\gamma\cos\theta} = 2\pi I_0(2\gamma)$ [6], where I_n is the modified Bessel function. States (12) minimize the first uncertainty relation in (8) and describe states with a nonzero energy spread. In fact, they reduce, for γ=0, to the eigenstates $(1/\sqrt{2\pi})\exp(im\theta)$ of p , whereas for large values of γ, $\psi_{min} \propto \exp(-\gamma\theta^2/2 + im\theta)$, i.e. the minimum uncertainty state becomes a Gaussian wavepacket with $\Delta p = \sqrt{\gamma/2}$ and $\Delta\theta = 1/\sqrt{2\gamma}$ such that $\Delta p \Delta\theta = 1/2$.

In general, $\Delta p = \sqrt{(\gamma/2)(I_1(2\gamma)/I_0(2\gamma))}$ [6]. These states, originally introduced by Jackiw [9] to describe the phase of the photon, could be useful to describe the energy spread in the quantum description of FEL.

QUANTUM FIELD DESCRIPTION

An alternative description to the N-particle Hamiltonian model can be formulated in the second-quantization formalism, treating the electrons as not interacting bosons [10]. In this formulation, the N particles are described by a matter-field operator $\hat{\Psi}(\theta, \bar{z})$ obeying the bosonic equal-time commutation relation $\left[\hat{\Psi}(\theta), \hat{\Psi}^+(\theta')\right] = \delta(\theta - \theta')$ and the normalization condition $\int_0^{2\pi} d\theta \hat{\Psi}(\theta)\hat{\Psi}^+(\theta) = \hat{N}$. In this formulation, the second-quantizied Hamiltonian is

$$\hat{H} = \int_0^{2\pi} d\theta \hat{\Psi}^+(\theta) H\left(\theta, -i\frac{\partial}{\partial\theta}, a, a^+\right)\hat{\Psi}(\theta) \quad (13)$$

where H is the single-particle Hamiltonian defined in (1). The Heisenberg equation for $\hat{\psi}$ and a are:

$$\begin{aligned} i\frac{\partial\hat{\Psi}}{\partial\bar{z}} &= -i\left[\hat{\Psi}, \hat{H}\right] = -\frac{1}{2\bar{\rho}}\frac{\partial^2\hat{\Psi}}{\partial\theta^2} - ig\left(ae^{i\theta} - h.c.\right)\hat{\Psi} \\ \frac{da}{d\bar{z}} &= -i\left[a, \hat{H}\right] = g\int_0^{2\pi} d\theta \hat{\Psi}^+(\theta) e^{-i\theta}\hat{\Psi}(\theta) + i\delta a \end{aligned} \quad (14)$$

Then, expanding the matter-wave field in the momentum basis, $\hat{\Psi}(\theta) = \sum_n c_n u_n(\theta)$, where $u_n(\theta) = (1/\sqrt{2\pi})e^{in\theta}$ are the eigenfunctions of p with eigenvalue n and c_n are the annihilation operators for the state with eigenvalue n, with $\left[c_n, c_{n'}^+\right] = \delta_{n,n'}$ Eqs.(14) become

$$\frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}}c_n + g\left(a^+c_{n+1} - ac_{n-1}\right)$$
$$\frac{da}{d\bar{z}} = g\sum_{-\infty}^{+\infty} c_n c_{n-1}^+ + i\delta a \quad (15)$$

The semiclassical regime of Eqs.(15), in which a and c_n are treated as classical functions, has been investigated in [8]. A fully quantum treatment of the linear regime of Eq. (14) has been given in [7], considering the equilibrium state with no photons and all the electrons in the state with n=0 (i.e. $\langle a\rangle_0 = 0$ and $\langle c_0^+c_0\rangle_0 = N$) . Then, considering c_1, c_{-1} and a as fluctuation operators, we obtain the same quantum linear equations (6), in which the bunching and the momentum bunching operators are defined as $B = c_1 + c_{-1}^+$ and $P = \left(c_1 - c_{-1}^+\right)/2$. In this description the electrons have initially a definite value of momentum (i.e. p=0), so that they are unlocalized in position.

The dynamics of the system is that of three parametric coupled harmonic oscillators, $a_1 = c_{-1}$, $a_2 = c_1$ and $a_3 = a$, which obey commutation rules $\left[a_i, a_j\right] = 0$ and $\left[a_i, a_j^+\right] = \delta_{i,j}$ for i=1,2,3. Starting from vacuum state of the three modes, it has been demonstrated [7] that the state at $\bar{z}$ is

$$|\psi(\bar{z})\rangle = \frac{1}{\sqrt{1+\langle n_1\rangle}}\sum_{m,n=0}^{\infty}\alpha_1^m(\bar{z})\alpha_2^m(\bar{z})\sqrt{\frac{(m+n)!}{m!n!}}|m+n,n,m\rangle \quad (16)$$

where $|\alpha_{1,2}|^2 = \langle n_{3,2}\rangle / \left(1+\langle n_1\rangle\right)$ and $\langle n_i\rangle = \langle a_i^+a_i\rangle$ are the average occupation numbers. The state n_1 refers to electrons with negative recoil (decelerating), n_2 with positive recoil (accelerating) and n_3 is the photon number. Note that the occupation number of the mode 1 is given by the sum of the other two, as a consequence of the constant of motion $C = n_1 - n_2 - n_3$, with $\langle C\rangle = 0$ when the system starts from vacuum. For the state (16), the number variance is $\sigma_i^2 = \langle n_i\rangle(1+\langle n_i\rangle)$ [1,2,7] i.e. the statistics is that of a thermal state. The state (16) is three-mode entangled, i.e. the recoiling electrons and the emitted photons are entangled.

It can be seen that for $\bar{\rho} \le 1$ the electrons, initially in the momentum state $n = 0$, can populate only the lower momentum state $n = -1$, recoiling backward by $\hbar k$ when a photon is emitted. In this quantum regime the system behaves as a two-level system [4], described by the two operators c_0 and c_{-1}. In the linear regime, the average number of photons grows exponentially as $\langle n_3\rangle \approx \langle n_1\rangle \approx (1/4)\exp(\sqrt{\bar{\rho}}\bar{z})$ at resonance (i.e. for $\delta = 1/2\bar{\rho}$) and the maximum number of emitted photons is N.

In the quantum regime the state (16) reduces to the pure bipartite state with $\langle n_2\rangle \approx 0$:

$$|\psi(\bar{z})\rangle \approx \left(1/\sqrt{1+\langle n_1\rangle}\right)\sum_{m=0}^{\infty}\alpha_1^m(\bar{z})|m,0,m\rangle \quad (17)$$

where $|\alpha_1|^2 \approx \langle n_1\rangle / \left(1+\langle n_1\rangle\right)$. The state (17) is **maximally entangled** because the photon and the recoiling electron are generated in pairs.

CONCLUSIONS

In conclusion, we have revised the quantum linear theory of the N-particle free-electron laser, introducing properly symmetrized electron collective operators. This allows to obtain the correct cubic characteristic equation, showing the shift and the narrowing of the FEL resonance. We have shown that intrinsic quantum mechanical properties of the momentum and position operators imply a very general minimum uncertainty relation between energy spread and bunching, yielding a quantum limitation to the maximum bunching which can be obtained in an FEL. A minimum uncertainty state has been properly defined so that it reduces to a gaussian packet in the small fluctuation limit. Using a second-quantized treatment we show that, in the quantum regime, $\bar{\rho} < 1$, the photon field and recoiling electrons are described by an maximally entangled quantum state. This property is well known to be quite fundamental for quantum information and quantum computing.

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