CONTROL OF THE INTENSITY OF A WAVE INTERACTING WITH CHARGED PARTICLES

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Abstract

The interaction of a wave with a beam of particles is of paramount importance in a great number of physical applications. We here focus on the case of a Free Electron Laser and review two control strategies aimed at re-shaping the inner topology of the single-particle phase-space to stabilize the oscillations of the laser intensity in the deep saturated regime.

INTRODUCTION

The interaction between a wave and a bunch of charged particles plays a central role in many branches of applied physics ranging from particle accelerators to laser physics. Generically, this self-consistent interaction leads to an exponential increase of the intensity of the wave, followed by an oscillating saturation. Oscillations are generated by the rotations in phase space of a clustered bunch of particles.

The wave-particle interaction can be cast in a Hamiltonian form with $N+M$ degrees of freedom, where N and M are resprectively the number of charged particles and electromagnetic waves. Examples include the so called electron beam-plasma instability, a phenomenon of paramount importance in the wide realm of kinetic plasma turbulence, and single-pass high-gain Free Electron Lasers (FELs). In the following we shall refer to the latter case, focusing in particular on seeding schemes where a small laser signal is injected at the entrance of the undulator and guides the subsequent amplification process [1]. Basic features of the system dynamics are successfully captured within a simplified one-dimensional framework discussed in the pioneering work by Bonifacio and collaborators [2]. The Hamiltonian reads:

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} - \delta I + 2\sqrt{\frac{I}{N}} \sum_{j=1}^{N} \sin(\theta_j - \varphi), \qquad (1)$$

where I and φ stands respectively for the intensity and phase of the wave, while the N conjugated pairs (p_j, θ_j) refer to the electrons. The detuning parameter δ measures the average relative deviation from the resonance condition.

As previously anticipated, the theory predicts a linear exponential instability and a late oscillating saturation for the

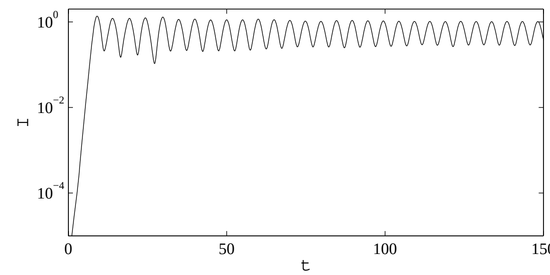

Figure 1: Normalized intensity calculated from the dynamics of Hamiltonian (1).

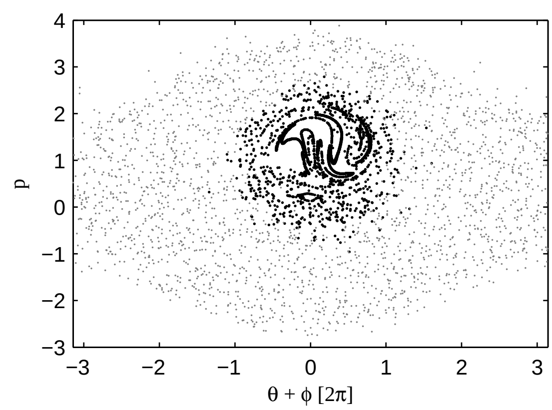

Figure 2: Snapshot of the N particles at $t = 1000$, with $N = 10^4$. The grey points correspond to the chaotic particles, the dark ones refer to the macro-particle

amplitude of the radiation field, Fig. 1. Inspection of the asymptotic phase-space, see Fig 2, suggests that a bunch of particles gets trapped in the resonance and forms a clump that evolves as a single macro-particle localized in space. The remaining particles are almost uniformly distributed between two oscillating boundaries, and populate the so called *chaotic sea* [3].

The macro-particle rotates around a well defined fixed point and this *microscopic* dynamics is shown to be responsible for the *macroscopic* oscillations observed at the intensity level. Qualitatively similar observations have been reported for the case of a Travelling Wave Tube (TWT)[4], a specially designed apparatus that mimics the plasma in-

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stability and enables to accurately investigate the non linear regime of the self-consistent wave-particles interaction. Given the above, it is an interesting problem to define dedicated strategies aiming at regularizing the saturated dynamics that could eventually contribute to improve the performance of the aforementioned devices.

The goal of this paper is to show that it is indeed possible to influence by an external perturbation the dynamics of the particles and enhance the stability of the system. To this end we shall consider a mean field type of approach which constitutes the natural reduction of the original N-body formulation (1). According to this simplified picture, the conjugated variables (ϕ, I) are replaced by two functions of time $\phi(t)$ and $I(t)$, the latter being obtained from direct simultations of the self-consistent dynamics. This in turn amounts to formally neglecting the action of the electrons on the field, an assumption that holds true in the late saturated regime.

The N-body Hamiltonian (1) can therefore be reduced to

$$\tilde{H_N} = \sum_{i=1}^{N} H_{1p}(\theta_i, p_i, t), \tag{2}$$

where

$$H_{1p}(\theta, p, t) = \frac{p^2}{2} - 2\sqrt{\frac{I(t)}{N}} \cos(\theta + \phi(t)). \tag{3}$$

In conclusion, the dynamics of a FEL can be adressed by monitoring the evolution of a *test particle*, obeying the Hamiltonian (3) where the functions $I(t)$ and $\phi(t)$ act as external fields and are here imposed by assuming their simplified asymptotic behaviour as obtained by a frequency analysis [5] :

$$2\sqrt{\frac{I(t)}{N}} e^{i\phi(t)} \approx F - \epsilon \sum_{k=1}^{K} W_k e^{i\omega_k t}, \tag{4}$$

in the reference frame of the wave.

TOWARD STABILIZATION: TWO ALTERNATIVE APPROACHES

Two different control strategies are here shortly discussed and shown to produce beneficial effects on stability of the system at saturation. For a detailed account on the techniques and an extensive report of the main findings, the interested reader may refer to [6] and [7].

Hamiltonian control of a test particle

First let us consider a Hamiltonian control technique. The method is based on the introduction of a small and apt modification of the potential that enables to recreate (alternatively destroy) invariant (KAM) tori in phase space. The Hamiltonian control addresses systems which are close to integrable, i.e. whose Hamiltonian can be written as $H = H_0 + V$, where H_0 is integrable and V a perturbation of order ϵ (compared to H_0). The results we use here have been proven rigorously [8, 9]. In practice, it can be shown that a suitable *control term* f of order ϵ^2 exists such that $H_0 + V + f$ has an invariant torus at a given frequency ω_0. In the present case, and focusing on the late saturated regime, the perturbation term is associated with the oscillating part of the intensity.

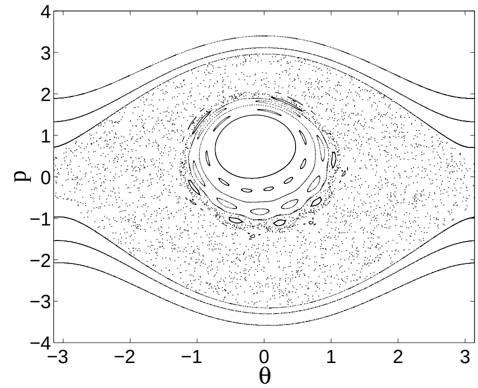

Figure 3: Poincaré sections of a test-particle of Hamiltonian $H_{1p}(\theta, p, t)$.

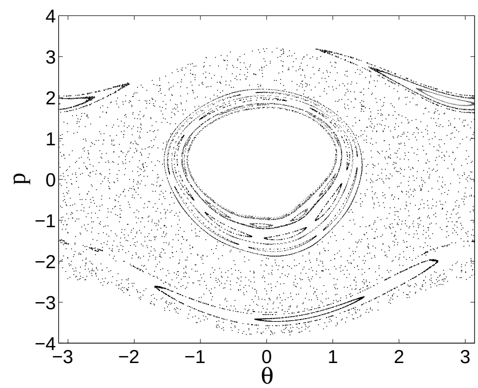

Figure 4: Poincaré sections of a *controlled* test-particle of Hamiltonian $H_{1p}(\theta, p, t) + \tilde{f}(\theta, p, t)$ (right).

The calculation of the control term [6] is carried on into action-angle variables (φ, J) and the derivation is not esplicitly reported here due to space limitations [1]. Instead, we shall present results of numerical experiments which clearly demonstrate (see Figs.3 and 4), that, in (θ, p) variables the analytical control term derived in [6] is successful in reconstructing some invariant tori around the macro-particle. In other words, it enlarges the macro-particle which in turn corresponds to enhancing the bunching factor, a quantity of paramount importance in FEL context. Finally, it is worth emphasising that according to this approach the form of the control term is derived and not imposed a priori.

[1] As a side remark, note that the exact change of variables from (φ, J) to (θ, p) presents a singularity at the pendulum separatrices. In order to implement our control on the whole space, a simplified, but regular, change of variables is used which mimics the exact one in the region of the invariant torus predicted by the control

The residue method

An alternative strategy can be elaborated that enables to modifying the intrinsic characteristics of the macro-particle. Contrary to the above technique, a (generic) parametrized perturbation is here a priori introduced, which allows to modify the topology of phase-space, by tuning appropriately the parameters. The residue method [10, 11, 12, 13] is used to predict the resulting local bifurcations, by an analysis of linear stability of periodic orbits. Information on the nature of these orbits (elliptic, hyperbolic or parabolic) is provided using e.g. an indicator like Greene's residue [10, 14], to monitor local changes of stability in a system subjected to an external perturbation [11, 12, 13, 15]. As we shall see, this approach enables one to tune the size, gyration radius and internal structure of the macroparticle, thus resulting in an effective tool for the stabilization of the intensity.

Consider an autonomous Hamiltonian flow with two degrees of freedom which depends on a set of parameters[2] $\boldsymbol{\lambda} \in \mathbb{R}^m$:

$$\dot{z} = \mathbb{J}\nabla H(z;\boldsymbol{\lambda}),$$

where $z = (p, E, \theta, t) \in \mathbb{R}^4$ and $\mathbb{J} = \begin{pmatrix} 0 & -\mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}$, and $\mathbb{I}_2$ being the two-dimensional identity matrix. In order to analyze the linear stability properties of the associated periodic orbits, we also consider the tangent flow written as

$$\frac{d}{dt}J^t(z) = \mathbb{J}\nabla^2 H(z;\boldsymbol{\lambda})J^t,$$

where $J^0 = \mathbb{I}_4$ and $\nabla^2 H$ is the Hessian matrix (composed of second derivatives of H with respect to its canonical variables). For a given periodic orbit with period T, the linear stability properties are given by the spectrum of the monodromy matrix J^T. These properties can be synthetically enclosed in the definition of Greene's residue :

$$R = \frac{4 - \mathrm{tr}J^T}{4}.$$

In particular, if $R \in]0, 1[$, the periodic orbit is elliptic; if $R < 0$ or $R > 1$ it is hyperbolic; and if $R = 0$ and $R = 1$, it is parabolic.

Since the periodic orbit and its stability depend on the set of parameters $\boldsymbol{\lambda}$, the features of the dynamics will change under apposite variations of such parameters. Generically, stability od periodic orbits is robust to small changes of parameters, except at specific values when bifurcations occur. The residue method [11, 12, 13] detects these rare events thus allowing one to calculate the appropriate values of the parameters leading to the prescribed behaviour in the dynamics. This method yield reduction as well as enhancement of chaos.

To illustrate the potentiality of method we shall introduce a parametrized perturbation, in the form [7]:

[2] At this level $\boldsymbol{\lambda}$ represents any generic family of parameters that influence the dynamics of the system

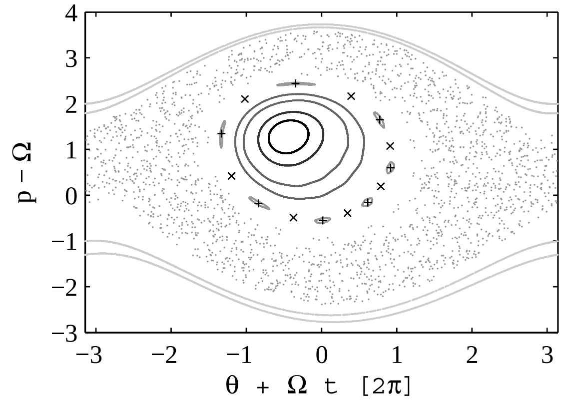

Figure 5: Poincaré section of a test-particle, described by Hamiltonian (3). The periodic orbits with r.n. 7 are marked by plus (elliptic orbit) and crosses (hyperbolic orbit). Ω stands for the wave velocity.

$$H^c_{1p}(\theta, p, t; \lambda) = H_{1p}(\theta, p, t) - 2\lambda\sqrt{\frac{I(t)}{N}}\cos(2\theta + \phi(t)). \quad (5)$$

Here λ controls the amplitude of the injected wave. Focus then on $\lambda = 0$, which corresponds to the original Hamiltonian H_{1p}, and consider two coupled Birkhoff periodic orbits, i.e. orbits having the same action but different angles in the integrable case and having the same rotation number (r.n.) on the Poincaré section, one elliptic $\mathcal{O}_e$ and one hyperbolic $\mathcal{O}_h$ (see Fig. 5) [3].

Call R_e and R_h the residues of these orbits: We have $R_e(0) > 0$ and $R_h(0) < 0$. We then modify the parameter λ until the following condition is matched :

$$R_e(\lambda_c) = R_h(\lambda_c) = 0, \quad (6)$$

at $\lambda_c = -0.0370$ [7]. Bifurcation (6) is associated with the creation of an invariant torus [13]. This diagnostic is confirmed by the Poincar section (see Fig.6) of the *controlled* Hamiltonian (5), at $\lambda = \lambda_c$: The elliptic islands with r.n. 7 have been replaced by a set of invariant tori, leading to an enlargement of the macro-particle. Note that elliptic islands with r.n. 6 are now present around the regular core.

The associated couple of elliptic/hyperbolic orbits can be treated similarly as those of r.n. 7, in order to gain further enlargement of the macro-particle (not reported here).

The control is naturally introduced in the self-consistent dynamics as :

$$H^c_N(I, \phi, p_i, \theta_i, \lambda) = H_N(I, \phi, p_i, \theta_i) - 2\lambda\sqrt{\frac{I}{N}}\sum_i \cos(2\theta_i + \phi) \quad (7)$$

[3] Let us recall that the rotation number (or winding number) of a periodic orbit is the number of times it crosses the Poincaré section before closing back on itself

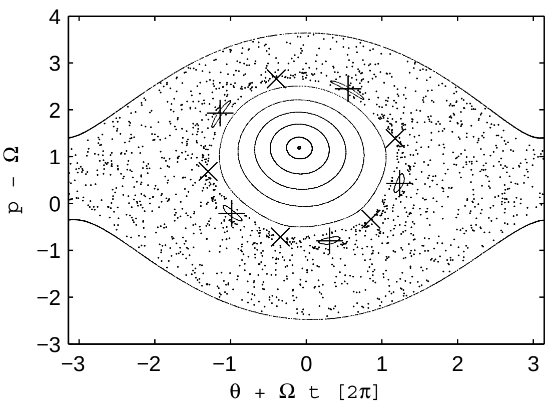

Figure 6: Poincaré section of a controlled test-particle of Hamiltonian (5), with $\lambda = \lambda_c \approx -0.0370$. The periodic orbits with r.n. 6 are marked by plus (elliptic orbit) and crosses (hyperbolic orbit). Ω stands for the wave velocity.

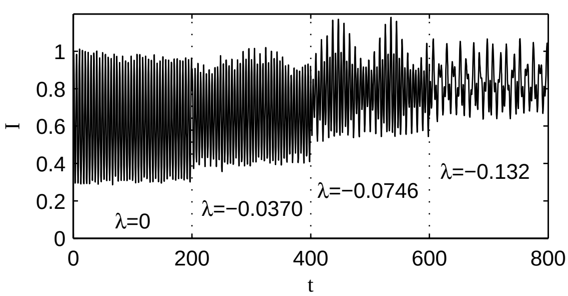

Figure 7: Intensity of the wave at saturation, in the four regimes : $\lambda = 0, -0.037, -0.0746$ and -0.1321 . Further improvements over this condition are discussed in [7]

where H_N is given by Eq. 1. The behaviour of the system is investigated in correspondence of the critical values $(\lambda_c, \lambda'_c, \lambda''_c)$ which have been calculated in the framework of the test-particle model, and identifying successive corrections as outlined in the precedding discussion. Importantly, the macro-particle is shown to increase also when operating within the relevant self-consistent context. As concerns the wave, the control results in a stabilization of its intensity (see Fig.7) [7].

CONCLUSIONS

In this paper, we focus on the physics of the wave-particle interaction and consider in particular the case of a FEL. Control techniques are developed in the framework of a simplified mean-field description aiming at stabilizing the laser behavior at saturation. In both cases, the size of the macro-particle is shown to be increased by adding a small pertubation to the system. These procedures result in a low–cost correction in term of energy[4]. Both approaches are utterly general and could be eventually considered to define innovative strategies aimed at adjusting the size of the macro-particle, and consequentely enhancing the bunching factor. In this respect, it is worth stressing that an experimental test of the Hamiltonian control method on a modified Travelling Wave Tube has been performed [16] in absence of self-consistency. Exploring the possibility of experimentally implementing the above control terms in both FEL and TWT contexts will be addressed in future research.

[4] As concerns the method of residues, the fact that the applied corrections are indeed small is confirmed by the values of λ that are calculated for the various setting considered above.

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