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PERIOD-AVERAGED SYMPLECTIC MAPS FOR THE FEL HAMILTONIAN*

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Abstract

Conventional treatments of synchrotron radiation in electron beams treat the radiation as a non-Hamiltonian aspect to the beam dynamics. However, the radiation can be modeled with an electromagnetic Hamiltonian. We present a period-averaged treatment of the FEL problem which includes the Hamiltonian aspects of the coupled electron-radiation dynamics. This approach is then applied to two problems: a 3D split-operator symplectic integrator, and a 1D single-mode FEL treated using Hamiltonian perturbation theory.

SYMPLECTIC MAP TREATMENT

Symplectic maps are useful for computing invariants in Hamiltonian systems and deriving symplectic integration schemes (among others) in single- or few-particle systems. Recent work has highlighted their use for studying many-body systems and self-consistent electromagnetic algorithms. Maps can also be applied to the period-averaged free-electron laser problem, using the factored map formalism and a first order Magnus expansion.

We begin with the Lagrangian for a system of relativistic electrons in a mix of external and self-consistent electromagnetic fields [1–3]:

$$\begin{aligned}\mathcal{L} = \sum_j -mc^2\sqrt{1-\left(\frac{\dot{\mathbf{x}}_j}{c}\right)^2} - e\phi(\mathbf{x}_j) + \frac{e}{c}\dot{\mathbf{x}}_j\cdot\mathbf{A}(\mathbf{x}_j) \\ + \frac{1}{8\pi}\int d\mathbf{x}\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t} - \nabla\phi\right)^2 - (\nabla\times\mathbf{A})^2.\end{aligned} \tag{1}$$

It is convenient to use s, the longitudinal variable, as the independent variable. We can do this by noting that the action integral $\mathcal{A} = \int dt\mathcal{L}$, and that $dt = (dt/ds)ds$ is a valid transformation of the integral as long as $ds/dt \neq 0$. This allows us to change the independent variable to the s-dependent Lagrangian $\mathcal{S}$ as

$$\begin{aligned}\mathcal{S} = \sum_j -mc^2\sqrt{\left(\frac{1}{c}\frac{d\tau_j}{ds}\right)^2 - \left(\frac{(\mathbf{x}_\perp)'_j}{c}\right)^2} \\ + \frac{e}{c}\tau'_j\phi(\mathbf{x}_j) + \frac{e}{c}(\mathbf{x}'_j)_\perp\cdot\mathbf{A}_\perp(\mathbf{x}_j) + \frac{e}{c}A_s(\mathbf{x}_j) \\ - \frac{1}{8\pi}\frac{1}{c}\int d\mathbf{x}_\perp d\tau\left(\frac{\partial\mathbf{A}}{\partial\tau} + \nabla\phi\right)^2 - (\nabla\times\mathbf{A})^2\end{aligned} \tag{2}$$

where the prime denotes total differentiation with respect to s and we have defined $\tau = -ct$ for dimensional convenience, so all the generalized coordinates have the same units. The action integral remains unchanged.

The scalar potential comes purely from self-consistent source terms – there are no electrostatic elements in our beamline – and $\mathbf{A}$ can be broken into external and self-consistent components. We need not worry about the dynamics of the external vector potentials. For simplicity, we can make the choice that $A_s^{(sc)} = 0$ (equivalent to the Weyl gauge $\phi = 0$ when we use t as the independent variable) and assume that $\phi = 0$ which neglects space charge effects. This leaves $\mathbf{A}_\perp = \mathbf{A}_\perp^{(ext.)} + \mathbf{A}_r$ and $A_s = A_s^{(ext.)}$, which captures the undulator fields and any external focusing elements like quadrupoles or dipoles, as well as the self-consistent radiation field.

We write the radiation field as

$$\mathbf{A}_r = \frac{mc}{e}\mathbf{e}_p\sum_\sigma u_\sigma e^{i\mathbf{k}_\perp^{(\sigma)}\cdot\mathbf{x}_\perp + ik_0^{(\sigma)}\tau} + c.c. \tag{3}$$

for a fixed, generally complex, polarization vector $\mathbf{e}_p = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}}$ with unit norm $|p_x|^2 + |p_y|^2 = 1$, and a range of perpendicular k-vectors and τ-components. The individual $u_\sigma(s)$ give the complex amplitude of a given radiation mode as a function of s. This gives the Lagrangian in terms of the individual mode amplitudes for the radiation, the external fields, and the particles to be:

$$\begin{aligned}\mathcal{S} = \sum_j -mc\sqrt{\left(\tau'_j\right)^2 - \left((\mathbf{x}'_\perp)_j\right)^2} \\ + \frac{e}{c}(\mathbf{x}'_\perp)_j\cdot(\mathbf{A}_\perp^{(ext.)}(\mathbf{x}_j) + \mathbf{A}_r) + \frac{e}{c}A_s(\mathbf{x}_j) \\ - \frac{1}{8\pi}\frac{1}{c}\left(\frac{mc}{e}\right)^2 \\ \sum_\sigma\left((k_0^{(\sigma)})^2 - |p_y k_x^{(\sigma)} - p_x k_y^{(\sigma)}|^2\right)|u_\sigma|^2 - |u'_\sigma|^2\end{aligned} \tag{4}$$

which then gives the canonical momenta for the electrons as well as for the individual modes as:

$$\begin{aligned}p_\tau &= \frac{mc\tau'}{\sqrt{\tau'^2 - 1 - (\mathbf{x}'_\perp)^2}}, \\ \mathbf{p}_\perp &= \frac{mc\mathbf{x}'_\perp}{\sqrt{\tau'^2 - 1 - (\mathbf{x}'_\perp)^2}} - \frac{e}{c}(\mathbf{A}_\perp^{(ext.)}(\mathbf{x}_j) + \mathbf{A}_r)\end{aligned} \tag{5}$$

and

$$\mathcal{P}_\sigma = -\frac{1}{4\pi c}\left(\frac{mc}{e}\right)^2 u'^*_\sigma \tag{6}$$

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for each individual particle and mode.

To compute symplectic maps, we must compute the Hamiltonian for this system, which is taken through the usual Legendre transformation over the j particle indices and the σ mode indices. The resulting Hamiltonian is given by:

$$\begin{aligned}\mathcal{H} = &-\sum_j \sqrt{\left(p_\tau^{(j)}\right)^2 - \left(\left(\mathbf{p}_\perp^{(j)} - \frac{e}{c}(\mathbf{A}_\perp^{(ext.)}(\mathbf{x}_j) + \mathbf{A}_r)\right)^2 - m^2c^2} \\ &+ \frac{e}{c}A_s(\mathbf{x}_j) + \sum_\sigma \frac{1}{2}\frac{4\pi}{c}\left(\frac{e}{mc}\right)^2 |\mathcal{P}_\sigma|^2 \\ &+ \frac{1}{2}\frac{c}{4\pi}\left(\frac{mc}{e}\right)^2 \left((k_0^{(\sigma)})^2 - |p_y k_x^{(\sigma)} - p_x k_y^{(\sigma)}|^2\right)|u_\sigma|^2. \end{aligned} \tag{7}$$

Assuming that p_τ is the dominant momentum component, which it usually is, we can Taylor expand the radical in powers of $1/p_\tau$ to get the approximate Hamiltonian for high energy electrons:

$$\begin{aligned}\mathcal{H} \approx &\sum_j -p_\tau^{(j)} + \frac{1}{2}\frac{m^2c^2}{p_\tau^{(j)}} + \frac{1}{2}\frac{\left((\mathbf{p}_\perp^{(j)} - \frac{e}{c}(\mathbf{A}_\perp^{(ext.)}(\mathbf{x}_j) + \mathbf{A}_r)\right)^2}{p_\tau^{(j)}} + \\ &\frac{e}{c}A_s + \sum_\sigma \frac{1}{2}\frac{4\pi}{c}\left(\frac{e}{mc}\right)^2 |\mathcal{P}_\sigma|^2 + \frac{1}{2}\frac{c}{4\pi}\left(\frac{mc}{e}\right)^2 \Omega_\sigma^2 |u_\sigma|^2,\end{aligned} \tag{8}$$

with $\Omega_\sigma^2 = (k_0^{(\sigma)})^2 - |p_y k_x^{(\sigma)} - p_x k_y^{(\sigma)}|^2$ being the natural frequency of the σ mode.

We can then break up $\mathbf{A}_\perp^{(ext.)}$ into the on-axis wiggler field and the off-axis wiggler field as

$$\mathbf{A}_\perp^{(ext.)} = \mathbf{A}_w(s) + \mathbf{A}_f(x, y, s), \tag{9}$$

which allows us to expand the perpendicular momentum term and break the Hamiltonian into the sum of the one-dimensional FEL Hamiltonian and the transverse focusing Hamiltonian.

The transverse canonical momentum term expands to

$$\begin{aligned}\left(\mathbf{p}_\perp - \frac{e}{c}(\mathbf{A}_w + \mathbf{A}_f + \mathbf{A}_r)\right)^2 = &\left(\mathbf{p}_\perp - \frac{e}{c}\mathbf{A}_f\right)^2 + \\ &\left(\mathbf{p}_\perp - \frac{e}{c}\mathbf{A}_f\right)\cdot(\mathbf{A}_w + \mathbf{A}_r) + \\ &|\mathbf{A}_w|^2 + 2\mathbf{A}_w\cdot\mathbf{A}_r + |\mathbf{A}_r|^2.\end{aligned} \tag{10}$$

The first term represents the finite transverse emittance dynamics and the undulator focusing terms. The second term is the dot product of the average transverse velocity with the undulator and radiation fields, and averages to zero. $|\mathbf{A}_r|^2$ will introduce the ponderomotive force on the average transverse particle motion, and this is also negligible.

Dropping these negligible terms leaves the FEL Hamiltonian as

$$\begin{aligned}\mathcal{H} \approx &\sum_j -p_\tau^{(j)} + \frac{1}{2}\frac{m^2c^2}{p_\tau^{(j)}} + \\ &\frac{1}{2}\frac{\left(\mathbf{p}_\perp^{(j)} - \frac{e}{c}\mathbf{A}_f\right)^2 + (e/c)^2|\mathbf{A}_w|^2 + 2(e/c)^2\mathbf{A}_w\cdot\mathbf{A}_r}{p_\tau^{(j)}} + \\ &\frac{e}{c}A_s + \sum_\sigma \frac{1}{2}\frac{4\pi}{c}\left(\frac{e}{mc}\right)^2 |\mathcal{P}_\sigma|^2 + \frac{1}{2}\frac{c}{4\pi}\left(\frac{mc}{e}\right)^2 \Omega_\sigma^2 |u_\sigma|^2.\end{aligned} \tag{11}$$

This is the Hamiltonian for relativistic particles in a radiation field and an undulator, with potential external focusing forces. At this point, more approximations are possible, such as linearizing $p_\tau^{(j)} = -\gamma_0 mc - \delta^{(j)}$ with $\delta \ll \gamma_0 mc$. We can also break the Hamiltonian up, for formal convenience, into $\mathcal{H}_0$, $\mathcal{H}_\perp$ and $\mathcal{V}_I$ for the electromagnetic, longitudinal, and transverse dynamics, respectively:

$$\begin{aligned}\mathcal{H}_0 = &\sum_j -p_\tau^{(j)} + \frac{1}{2}\frac{m^2c^2}{p_\tau^{(j)}} + \frac{1}{2}\left(\frac{e}{c}\right)^2\frac{|\mathbf{A}_w|^2}{p_\tau^{(j)}} + \\ &\sum_\sigma \frac{1}{2}\frac{4\pi}{c}\left(\frac{e}{mc}\right)^2 |\mathcal{P}_\sigma|^2 + \frac{1}{2}\frac{c}{4\pi}\left(\frac{mc}{e}\right)^2 \Omega_\sigma^2 |u_\sigma|^2, \\ \mathcal{H}_\perp = &\frac{1}{2}\sum_j \frac{\left(\mathbf{p}_\perp^{(j)} - \frac{e}{c}\mathbf{A}_f\right)^2}{p_\tau^{(j)}} + \frac{e}{c}A_s, \\ &\text{and} \\ \mathcal{V}_I = &2\left(\frac{e}{c}\right)^2\frac{\mathbf{A}_w\cdot\mathbf{A}_r}{p_\tau^{(j)}}.\end{aligned} \tag{12}$$

The combination $\mathcal{H}_0 + \mathcal{V}_I$, if we assume $\mathbf{k}_\perp^{(\sigma)} = 0$, is the one-dimensional FEL Hamiltonian. $\mathcal{H}_\perp$ captures the transverse dynamics, assuming as we have that there is no space charge – if we were to include space charge in this treatment there would be a self-consistent A_s or ϕ and these would also contribute to the transverse and longitudinal dynamics.

We are now in a position to compute symplectic maps over a single wiggler period, $\mathcal{M}_{s\to s+l_w}$.

The symplectic map satisfies the operator differential equation

$$\mathcal{M}' = \mathcal{M} :-\mathcal{H}: \tag{13}$$

where $:-\mathcal{H}:$ is the Hamiltonian Lie operator [4–6], which generates s transformations on the particle-field coupled phase space. There are multiple applications to this map: (1) we can compute a map for the one-dimensional FEL problem that includes the field terms; (2) we can derive symplectic integrators for the 1D and 3D FEL Hamiltonians, especially for tapering and other more schemes; (3) we can compute invariants for the 1D (and, less likely, but possibly) the 3D FEL Hamiltonians.

For the 1D FEL problem, we have that

$$\mathcal{M}_{1D}' = \mathcal{M}_{1D} :-\mathcal{H}_0 - \mathcal{V}_I: \tag{14}$$

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where $\mathcal{H}_0$ is exactly integrable – it is the radiation modes as harmonic oscillators with mass $m\sqrt{c}/4\pi e$ and frequency Ω_σ and motion in a drift where $|\mathbf{A}_w|^2$ may have some s-dependence for a planar undulator, for example.

Using the factored map formalism, we can write $\mathcal{M}_{1D} = \mathcal{M}_{1D}^{(I)}\mathcal{M}_{1D}^{(0)}$, where

$$(\mathcal{M}_{1D}^{(0)})' = \mathcal{M}_{1D}^{(0)} :-\mathcal{H}_0: \tag{15}$$

and

$$(\mathcal{M}_{1D}^{(I)})' = \mathcal{M}_{1D}^{(I)} \left(\mathcal{M}_{1D}^{(0)} :-\mathcal{V}_I:(\mathcal{M}_{1D}^{(0)})^{-1}\right). \tag{16}$$

The transformation of $\mathcal{V}_I$ integrates along the unperturbed trajectory, as the similarity transformation on the Lie operator passes through the colons and gives that

$$\mathcal{M}_{1D}^{(0)} :-\mathcal{V}_I:(\mathcal{M}_{1D}^{(0)})^{-1} = :-\mathcal{M}_{1D}^{(0)}\mathcal{V}_I:. \tag{17}$$

The unperturbed trajectory ends up giving the transformed interaction potential as

$$\begin{aligned}\mathcal{M}_{1D}^{(0)}\mathcal{V}_I =& \frac{2e^2}{\gamma_0 mc^3}\mathbf{A}_w\times \\ & \mathbf{e}_p\frac{mc}{e}\sum_\sigma\sum_j\left(u_\sigma + \frac{e}{mc}\mathcal{P}_\sigma^*\right)e^{i\Omega^{(\sigma)}s}\times \\ & e^{i\mathbf{k}_\perp^{(\sigma)}\cdot\mathbf{x}_\perp^{(j)} + ik_0^{(\sigma)}(\tau^{(j)}-\psi^{(j)}(s))} + c.c.\end{aligned} \tag{18}$$

where

$$\psi(s) = -s - \frac{1}{2}\frac{m^2c^2}{(p_\tau^{(j)})^2}s - \frac{1}{2}\frac{m^2c^2}{(p_\tau^{(j)})^2}\left(\frac{e}{c}\right)^2\int_{s_0}^s ds'|\mathbf{A}_w|^2 \tag{19}$$

is the unperturbed drift trajectory.

To first order in a Magnus expansion [7, 8], we can compute the interaction map to be given by

$$\mathcal{M}_{1D}^{(I)} \approx \exp\left\{-\int_{s_0}^s ds' :\mathcal{M}_{1D}^{(0)}\mathcal{V}_I: + O(s^2)\right\}. \tag{20}$$

This begins to introduce the 1D FEL resonance condition, as the exponent of the map is proportional to just s if the resonant condition is satisfied. A full analysis of $\mathcal{M}_{1D}^{(I)}$ is beyond the scope of this proceeding, as it contains the entire FEL interaction including saturation, if it is taken to sufficiently long s.

We can also consider the 3D FEL problem numerically by using a split-operator approach. We can approximate the full map as the symmetric product of partial maps:

$$\mathcal{M}_{s_0\to s_0+l_2} \approx \mathcal{M}_{(s_0\to s_0+l_2)/2}^{(\perp)}\mathcal{M}_{1D}\mathcal{M}_{(s_0\to s_0+l+2)/2}^{(\perp)} \tag{21}$$

where $\mathcal{M}_{(s_0\to s_0+l_2)/2}^{(\perp)}$ integrates the perpendicular Hamiltonian weighted by a factor of a half from s_0 to $s_0 + l$. We can carry over the $1D$ map, and approximate the perpendicular map using the same first order Magnus expansion:

$$\begin{aligned}&\mathcal{M}_{(s_0\to s_0+l_w)/2}^{(\perp)} \approx \\ &\exp\left\{-\frac{1}{2}\int_{s_0}^{s_0+l_w} :\sum_j\frac{\left(\mathbf{p}_\perp^{(j)} - \frac{e}{c}\mathbf{A}_f\right)^2}{2p_\tau^{(j)}} + \frac{e}{c}A_s:\right\}.\end{aligned} \tag{22}$$

If we assume that A_s has no explicit s dependence over the range of integration and that $\int_{s_0}^{s_0+l_w}\mathbf{A}_f = 0$, which is the case if $\mathbf{A}_f$ is periodic with the wiggler period and l_2 is the undulator period, then this maps becomes

$$\begin{aligned}&\mathcal{M}_{(s_0\to s_0+l_2)/2}^{(\perp)} \approx \\ &\exp\left\{-\frac{1}{2}:\sum_j\frac{(\mathbf{p}_\perp^{(j)})^2 - \left(\frac{e}{c}\right)^2\langle\mathbf{A}_f^2\rangle}{2p_\tau^{(j)}} + \frac{e}{c}A_s: l_w\right\}\end{aligned} \tag{23}$$

where we have averaged $\mathbf{A}_f^2$ over the wiggler period. This map can then be split in half again as a drift-kick map, giving the correct trajectories to order l_w^3.

CONCLUSION

In this report, we have highlighted a derivation of symplectic maps as can be applied to the one-dimensional or three-dimensional free-electron laser problem. These maps could be applied to computing invariants and saturation effects in one-dimensional free-electron lasers or for computing second-order symplectic integrators for the three-dimensional free-electron laser. It remains to apply these approaches and determine the applicability of these symplectic maps.

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