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ARBITRARY ORDER PERTURBATION THEORY FOR A TIME-DISCRETE MODEL OF MICRO-BUNCHING DRIVEN BY LONGITUDINAL SPACE CHARGE

Ph. Amstutz*, Lawrence Berkeley National Lab, Berkeley, USA
M. Vogt, Deutsches Elektronen-Synchrotron DESY, Hamburg, Germany

Abstract

A well established model for studying the micro-bunching instability driven by longitudinal space charge in ultra-relativistic bunches in FEL-like beamlines can be identified as a time-discrete Vlasov system with general drift maps and Poisson type collective kick maps. Here we present an arbitrary order perturbative approach for the general system and the complete all-orders solution for a special example. For this example we benchmark our theory against our Perron-Frobenius tree-code.

INTRODUCTION

Longitudnial dynamics in the magnetic bunch-compressors of free-electron laser injectors can drive a micro-bunching instability, deteriorating the electron bunch quality [1, 2]. As established in References [3, 4] the space-charge driven micro-bunching effect – in the ultra-relativistic limit – be investigated by means of an time-discrete model. In this model the longitudinal dynamics of the system are exactly captured by a combination of drift maps $D[\lambda]\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$D[\lambda]\colon z = (q, p) \mapsto (q + \lambda(p), p)$$

with a dispersion function $\lambda \in C^1(\mathbb{R}, \mathbb{R})$ and kick maps $K[k]\colon \mathbb{R}^2 \to \mathbb{R}^2$

$$K[k]\colon (q, p) \mapsto (q, p + k(q))$$

with a kick function $k \in C^1(\mathbb{R}, \mathbb{R})$. The collective nature of the micro-bunching is accounted for by adopting the Vlasov-picture, where the electron bunch is described by its probability density in phase space $\Psi\colon \mathbb{R}^2 \to \mathbb{R}$. A remarkable property of the model is the fact that it remains exactly solvable in terms of time-discrete maps, even in presence of *collective* kick functions $k[\Psi]$. It can be shown that a phase-space density (PSD) evolves as

$$\Psi(x) \mapsto \Psi(M^{-1}(x))$$

where M is the solution of the equations of motion of the system. A PSD is therefore propagated by objects defined by

$$\mathcal{M}\Psi := \Psi \circ M^{-1}$$

called Perron-Frobenius operators (PFO). The overall effect of a single bunch-compressor stage is then given by

$$\mathcal{M}[\Psi] = \mathcal{D}_{\mathrm{Chic}}\, \mathcal{K}_{\mathrm{Cav}} \mathcal{K}[\Psi]_{\mathrm{SC}}$$

where $\mathcal{D}_{\mathrm{Chic}}$ is the PFO associated to the drift map describing the dispersive magnetic chicane. $\mathcal{K}_{\mathrm{Cav}}$ and $\mathcal{K}_{\mathrm{SC}}$ are the PFOs of the kick maps describing the impact of the accelerating cavity and the collective self-interaction of the bunch in the field-free region upstream of the chicane, respectively. Thus the complete time-discrete Vlasov system is defined by

$$\Psi \mapsto \mathcal{M}[\Psi]\,\Psi\,.$$

PERTURBATION THEORY

The effect that initially small perturbations $\varepsilon\phi$ of an otherwise homogeneous charge distribution Ψ amplify themselves by virtue of a collective self-interaction

$$\mathcal{M}[\Psi + \varepsilon\phi](\Psi + \varepsilon\phi) \tag{1}$$

is generally referred to as *Micro-bunching*. While PFOs are linear operators in the sense that $\mathcal{M}(c_1\Psi_1 + c_2\Psi_2) = c_1\mathcal{M}\Psi_1 + c_2\mathcal{M}\Psi_2$, collective PFOs are generally not linear in their *internal* dependence on the PSD $\mathcal{M}[c_1\Psi_1 + c_2\Psi_2] \neq c_1\mathcal{M}[\Psi_1]+c_2\mathcal{M}[\Psi_2]$. This drives the need for a perturbation theory to investigate Equation (1), as derived in [4, 5]. It has been shown that Equation (1) for a general collective PFO $\mathcal{M}[\Psi]$ can be expanded in ε

$$\begin{aligned} \mathcal{M}[\Psi_0 + \varepsilon\phi_0](\Psi + \varepsilon\phi_0) &= \mathcal{M}[\Psi_0]\Psi_0 + o(\varepsilon^{N+1}) \\ &+ \sum_{n=1}^{N} \varepsilon^n \left(\mathcal{M}^{(n)}[\Psi_0]\phi_0^n\Psi_0 + \mathcal{M}^{(n-1)}[\Psi_0]\phi_0^n\right) \\ &\equiv \Psi_1 + \sum_{n=1}^{N} \varepsilon^n \phi_{1,n} + o(\varepsilon^{N+1}), \end{aligned} \tag{2}$$

where $\mathcal{M}^{(n)}$ is the n-th Frechet derivative of $\mathcal{M}$. $\Psi_1 \equiv \mathcal{M}[\Psi_0]\Psi_0$ is the solution of the unperturbed system.

For a collective PFO of the form

$$\mathcal{M}[\Psi] = \mathcal{L}\mathcal{K}[k[\Psi]] \tag{3}$$

with an arbitrary non-collective PFO $\mathcal{L}$ it can be shown that the n-th order propagated perturbations take the form

$$\begin{aligned} \phi_{1,n} = (-1)^n \mathcal{M}[\Psi_0]\Big(&\frac{k[\phi_0]^n \circ Q}{n!}\,\partial_p^n \Psi_0 \\ &-\frac{k[\phi_0]^{n-1} \circ Q}{(n-1)!}\,\partial_p^{n-1}\phi_0\Big), \end{aligned} \tag{4}$$

where the projection operator $Q\colon \mathbb{R}^2 \to \mathbb{R}$

$$Q\colon (q.p) \mapsto q$$

has been introduced. For a detailed derivation see References [4, 5].

* amstutz@lbl.gov

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EXAMPLE: LONG BUNCH, HARMONIC MODULATION

Consider an long bunch with undisturbed PSD $\Psi_0(q,p) = \Lambda(\alpha;q)\xi(0,\sigma;p)$, where $\xi(\mu,\sigma;\cdot)$ is a Gaussian distribution and

$$\Lambda(\alpha;q) \equiv \begin{cases} 1/\alpha & -\alpha/2 \le q \le \alpha/2 \\ 0 & |q| \ge \alpha/2 + d \\ \text{smooth} & \alpha/2 \le |q| \le \alpha/2 + d \end{cases} \tag{5}$$

with $d \ll \alpha$ and α large enough to make edge-effects negligible. Let

$$\mathcal{M}[\Psi] = \mathcal{D}[q \mapsto \beta q]\mathcal{K}[k[\Psi]], \tag{6}$$

$k[\Psi] = \mathcal{G} * \int_{\mathbb{R}} \Psi \mathrm{d}p$ being a Poisson-type collective kick with Greens function $\mathcal{G}$. This setup corresponds to a single bunch compressor stage with vanishing RF-Voltage and therefore no actual compression. For disturbances of the form $\varepsilon\phi_0(q,p) = \Psi_0(q,p)S(q)$ it can be shown that the propagated disturbances (4) take the form

$$\begin{aligned}\phi_{1,n} =& (-1)^n \Lambda(\alpha;q)\xi(0,\sigma;p) \\ &\Big(\frac{k[\phi_0]^n(q-\beta p)}{n!}\tilde{H}_n(\sigma;p) \\ &-\frac{k[\phi_0]^{n-1}(q-\beta p)}{(n-1)!}S(q-\beta p)\tilde{H}_{n-1}(\sigma;p)\Big),\end{aligned} \tag{7}$$

where $\tilde{H}_n(\sigma;p) \equiv H_n(p/\sqrt{2\sigma^2})(-\sqrt{2\sigma^2})^{-n}$ the modified n-th Hermite polynomial.

For a sinusoidal modulation $S(q) \equiv \mu\sin(\kappa q)$ the collective kick function can be solved explicitly

$$k[\phi_0](q) = \hat{\mu}_\kappa \cos(\kappa q),$$

with $\hat{\mu}_\kappa \equiv \tau\mu\Im\tilde{\mathcal{G}}(\kappa)/\alpha$, where τ is proportional to the length of the chicane and $\tilde{\mathcal{G}}(\kappa)$ is the Fourier transform of the Greens function (impedance). Of particular interest for the investigation of microbunching effects is the evolution of the spatial charge densities of the propagated disturbance terms $\rho_{1,n}(q) = \int_{\mathbb{R}} \phi_{1,n}$, which for the example at hand can be calculated explicitly, albeit with significant algebraic effort as carried out in [5]. Depending on the evenness of n two different terms are obtained

$$\begin{aligned}\rho_{1,n,\text{even}} =& \frac{(-1)^{\frac{n}{2}}\hat{\mu}_\kappa^n\beta^n\kappa^n}{n!\,2^{n-1}\,\alpha}\Big\{ \sum_{k=1}^{\frac{n}{2}}\Big(\\ &\binom{n}{\frac{n}{2}-k}\cos(2k\kappa q)(2k)^n \mathrm{e}^{-\frac{1}{2}(2k\sigma\beta\kappa)^2} \\ &+\frac{n\mu}{\hat{\mu}_\kappa\beta\kappa}\binom{n-1}{\frac{n}{2}-k}\Big[\cos(2k\kappa q)(2k)^{n-1}\mathrm{e}^{-\frac{1}{2}(2k\sigma\beta\kappa)^2} \\ &-\cos([2k-2]\kappa q)(2k-2)^{n-1}\mathrm{e}^{-\frac{1}{2}(\sigma\beta\kappa[2k-2])^2}\Big]\Big)\Big\}.\end{aligned} \tag{8}$$

$$\begin{aligned}\rho_{1,n,\text{odd}} =& \frac{(-1)^{\frac{n-1}{2}}\hat{\mu}_\kappa^n\beta^n\kappa^n}{n!\,2^{n-1}\,\alpha}\Big\{ \sum_{k=0}^{\frac{n-1}{2}}\Big(\\ &\binom{n}{\frac{n-1}{2}-k}\sin(\kappa q[2k+1])(2k+1)^n\mathrm{e}^{-\frac{1}{2}(\sigma\beta\kappa[2k+1])^2} \\ &+\frac{n\mu}{\hat{\mu}_\kappa\beta\kappa}\binom{n-1}{\frac{n-1}{2}-k}\Big[\sin(\kappa q[2k+1])(2k+1)^{n-1}\mathrm{e}^{-\frac{1}{2}(\sigma\beta\kappa[2k+1])^2} \\ &-\sin(\kappa q[2k-1])(2k-1)^{n-1}\mathrm{e}^{-\frac{1}{2}(\sigma\beta\kappa[2k-1])^2}\Big]\Big) \\ &-\frac{n\mu}{\hat{\mu}_\kappa\beta\kappa}\binom{n-1}{\frac{n-1}{2}}\sin(\kappa q)\mathrm{e}^{-\frac{1}{2}(\sigma\beta\kappa)^2}\Big\}.\end{aligned} \tag{9}$$

While we can report that direct numerical evaluation has shown that the sum $\sum_{n=1}^{N}\rho_{1,n}$ does converge (albeit slowly) for $N \longrightarrow \infty$ over a wide range of parameters, a formal investigation of its convergence properties is still outstanding.

SIMULATION

In order to verify the presented analytical results we compare them to simulations using our tree-based Perron Frobenius code [6]. The lower plots in Figures 1 show the propagated PSDs

$$\Psi_1 = \mathcal{D}[q \mapsto \beta q]\mathcal{K}[k[\Psi_0]]\Psi_0,$$

where the initial density – as above – is given by

$$\Psi_0(q,p) = \Lambda(\alpha;q)\xi(0,\sigma;p)[1+\mu\sin(\kappa q)].$$

For the impedance term, we chose a model that treats the longitudinal space-charge force as the force that a cylindrical bunch with radius a exerts on an electron on its axis

$$\Im\tilde{\mathcal{G}}(\kappa) = \frac{2}{a^2\,\kappa}\left[1 - a\,|\kappa|\,K_1(a\,|\kappa|)\right], \tag{10}$$

with the first-order modified Bessel-Function K_1. In the upper plots the spatial charge densities obtained from simulation (black) is compared to the analytical result (red) given by $\rho = \int_{\mathbb{R}} \Psi_1 + \sum_{n=1}^{N=128}\phi_{1,n}\mathrm{d}p$. All parameters are the same in both cases; their explicit value is not meaningful in the framework of this dimensionless benchmark. In the left plot the case of vanishing chicane strength $\beta = 0$ is depicted whereas in the right plot β is set to a positive value. It can be seen that in both cases the analytical and numerical results are in very good agreement.

The right plot shows the formation of “double horns” in the spatial charge density: Around this particular chicane setting $\beta^\star$, the crests of the momentum-modulated PSD overlap in such a way to form *two* maxima. For $\beta \ll \beta^\star$ only a single maximum resulting from the erecting falling edge of the modulation is formed. For $\beta \gg \beta^\star$ the bunching washes out and is much less pronounced. Investigation of this effect will be the subject of future work.

CONCLUSION

We have summarized recent advancements of our Perron-Frobenius-Vlasov perturbation-theory. For a specific, yet

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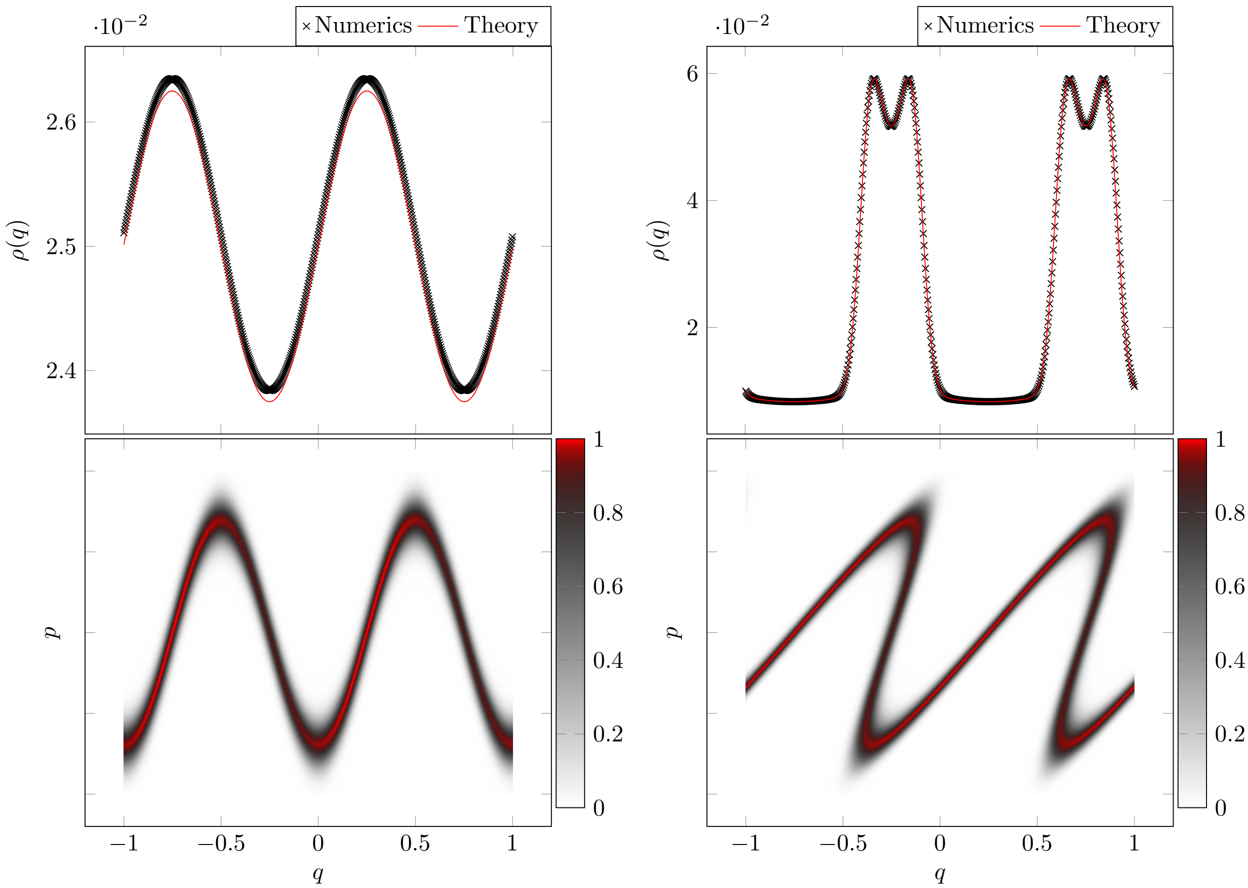

Figure 1: Comparison of the charge densities obtained from simulation with our Perron-Frobenius code and the analytical result (top plots), for vanishing (left) and a positive (right) chicane strength. The lower plots depict the PSD calculated by the code.

practically relevant example the theory is analytically tractable to arbitrary order and yields predictions for both, PSD and charge density of the propagated bunch, which have been presented. A cross-check between analytical and numerical results has been performed, showing good agreement.

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