ISBN: 978-3-95450-204-2 ISSN: 2673-5350 doi:10.18429/JACoW-IBIC2019-MOCO03

ESTIMATION OF LONGITUDINAL PROFILES OF ION BUNCHES IN THE LHC USING SCHOTTKY-BASED DIAGNOSTICS

K. Lasocha[1], D. Alves, CERN, Geneva, Switzerland
[1]also at Institute of Physics, Jagiellonian University, Kraków, Poland

Abstract

The Large Hadron Collider (LHC) Schottky monitors have been designed to measure various parameters of relevance to beam quality, namely tune, momentum spread and chromaticity. We present another application of this instrument - the evaluation of longitudinal bunch profiles. The relation between the distribution of synchrotron amplitudes within the bunch population and the longitudinal bunch profile is derived from probabilistic principles. Our approach, limited to bunched beams with no intra-bunch coherent motion, initially fits the cumulative power spectral density of acquired Schottky spectra with the underlying distribution of synchrotron amplitudes. The result of this fit is then used to reconstruct the bunch profile using the derived model. The obtained results are verified by a comparison with measurements from the LHC Wall Current Monitors.

INTRODUCTION

The Large Hadron Collider (LHC) transverse Schottky system, whose main objective is to provide the beam operators with non-invasive, bunch-by-bunch tune and chromaticity measurements was commissioned in 2011 [1]. In the meantime, the system has undergone major upgrades in order to improve signal quality [2]. Still, although qualitatively its chromaticity estimates seem to agree with trends from other measurement techniques (as verified in dedicated experiments), the quantitative discrepancies observed still need to be fully understood [3]. Studies are therefore underway in order to better understand the spectra obtained. will , under the assumption that there is no coherent intra-bunch motion. It should be noted, however, that the LHC Schottky monitors are designed for transverse measurements, and as such are not optimised for measurements in the longitudinal plane.

LONGITUDINAL BUNCH PROFILE

In hadron machines such as the LHC, where radiation losses are small, the RF phase difference, $\Delta\phi_{RF}$, between a given particle within the bunch and the synchronous particle obeys the pendulum equation [4, Eq. (9.51)]

$$\frac{d^2\Delta\phi_{RF}}{dt^2} + \omega_{s_0}^2 \sin(\Delta\phi_{RF}) = 0 \tag{1}$$

where ω_{s_0} is the nominal synchrotron frequency. For RF harmonic h, revolution frequency ω_0 and time amplitude (maximum time difference between a given particle and the synchronous particle) of synchrotron oscillations $\widehat{\tau}$, we have that the particle's synchrotron frequency is given by:

$$\omega_s = \frac{\pi}{2\mathcal{K}\left[\sin\left(\frac{h\omega_0\widehat{\tau}}{2}\right)\right]}\omega_{s_0}, \tag{2}$$

where $\widehat{\Delta\phi_{RF}} = h\omega_0\widehat{\tau}$ is the RF phase amplitude of synchrotron oscillations and $\mathcal{K}([0,1]) \to [\pi/2, \infty]$ is the complete elliptic integral of the first kind [5, p. 590]. This comes from the general theory of an arbitrary-amplitude pendulum [6].

From [7] we know that the time difference τ between a particle performing synchrotron motion and the synchronous particle is described by a simple harmonic motion, i.e.:

$$\tau = \tau(\widehat{\tau}, \phi_s) = \widehat{\tau}\cos(\omega_s t + \phi_s). \tag{3}$$

The longitudinal bunch profile can be interpreted as the probability distribution of τ. We shall denote this distribution by $\mathcal{B}(\tau)$. The assumption of no coherent intra-bunch motion implies that the distribution of initial synchrotron phases, ϕ_s, is uniform and independent of the distribution of synchrotron amplitudes $\widehat{\tau}$. Furthermore, under stationary conditions, the longitudinal bunch profile is independent of time. Therefore, the probability of finding a particle with time difference τ with respect to the synchronous particle depends only on its amplitude of oscillation $\widehat{\tau}$:

$$\mathcal{B}(\tau) = \int_0^\infty g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau})d\widehat{\tau} = \int_{|\tau|}^\infty g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau})d\widehat{\tau},$$

where $g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau})$ is the joint probability density of a particle having amplitude $\widehat{\tau}$ and time difference τ. The second equality comes from the fact, that $g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau}) = 0$ for $|\tau| > \widehat{\tau}$. Derivation of $g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau})$ is not straightforward, as τ and $\widehat{\tau}$ are not independent, but it can be derived from the joint distribution of initial synchrotron phases and amplitudes $g_{\phi_s,\widehat{\tau}}$. We can write

$$g_{\phi_s,\widehat{\tau}}(\phi_s,\widehat{\tau}) = g_{\phi_s}(\phi_s)\,g_{\widehat{\tau}}(\widehat{\tau}) = \frac{g_{\widehat{\tau}}(\widehat{\tau})}{2\pi},$$

as these random variables are independent and ϕ_s is uniformly distributed. In addition, let us define the transformation

$$u = (u_1, u_2) : (\phi_s, \widehat{\tau}) \mapsto (\tau, \widehat{\tau})$$

where u_1 is defined by Eq. (3) and u_2 is the identity function of $\widehat{\tau}$. The relationship between the joint distributions of two sets of random variables related by a known transformation function is given in [8, p. 201]. Using this, we obtain

$$g_{\tau,\widehat{\tau}}(\tau,\widehat{\tau}) = \frac{2g_{\phi_s,\widehat{\tau}}(\phi_s,\widehat{\tau})}{\sqrt{\widehat{\tau}^2 - \tau^2}} = \frac{g_{\widehat{\tau}}(\widehat{\tau})}{\pi\sqrt{\widehat{\tau}^2-\tau^2}}$$

ISBN: 978-3-95450-204-2 ISSN: 2673-5350 doi:10.18429/JACoW-IBIC2019-MOCO03

which finally enables us to write

$$\mathcal{B}(\tau) = \int_{|\tau|}^{\infty} \frac{g_{\widehat{\tau}}(\widehat{\tau})}{\pi\sqrt{\widehat{\tau}^2 - \tau^2}} d\widehat{\tau}. \quad (4)$$

Equation (4) allows us to calculate the longitudinal bunch profile knowing the distribution of synchrotron amplitudes. Conversely, if we are given the bunch profile, we can extract $g_{\widehat{\tau}}$ by numerically solving an integral equation [9]. In addition, as the synchrotron amplitudes are related to the synchrotron frequencies by Eq. (2), knowing one of these distributions allows us to determine the other two, as shown in Fig. 1.

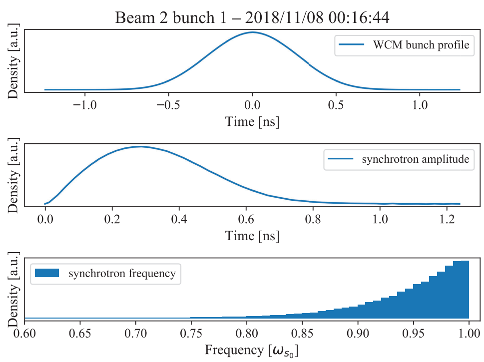

Figure 1: Top: typical bunch profile at flattop, measured by a Wall Current Monitor; Middle: synchrotron amplitude distribution derived from Eq. (4); Bottom: synchrotron frequency distribution derived from Eq. (2).

SCHOTTKY SPECTRUM

The Schottky spectrum in the vicinity of a revolution harmonic consists of three regions of interest: the longitudinal/central part and the two transverse sidebands. Each region is composed of a series of Bessel satellites (J_p) of finite width due to the presence of many particles with different synchrotron frequencies (see as example Fig. 4). In the scope of this paper we are only interested in the longitudinal part of the spectrum. The intensity signal due to a single-particle i, in the vicinity of the n-th revolution harmonic, can be written in the following form [7]

$$s_i(t) \propto \Re\left(\sum_{p=-\infty}^{\infty} j^p J_p\left(n\omega_0\widehat{\tau}_i\right) e^{j\left(n\omega_0 t + p\omega_{s_i} t + p\phi_{s_i}\right)}\right). \quad (5)$$

Let's now consider the Power Spectral Density (PSD), $P(\omega)$, of a pick-up signal $s(t) = \sum_i^N s_i(t)$, which is the sum of the individual contributions of N particles. By examining Eq. (5) we conclude that $P(\omega)$ is non-deterministic, but depends on the random synchrotron phases ϕ_{s_i}. We shall then start by deriving the expected value of the PSD

$$\langle P(\omega)\rangle = \langle F(\omega) F^*(\omega)\rangle, \quad (6)$$

where $F(\omega) = \int_{-\infty}^{\infty} s(t)e^{-j\omega t}dt$ is the Fourier transform of $s(t)$ and $F^*(\omega)$ is its complex conjugate. Since the Fourier transform is linear, the previous equation can also be written in the following form

$$\begin{aligned}\langle P(\omega)\rangle &= \left\langle\left[\sum_{i=1}^{N} F_i(\omega)\right] \times \left[\sum_{i=1}^{N} F_i^*(\omega)\right]\right\rangle = \\ &= \left\langle \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} F_i(\omega)F_j^*(\omega) + \sum_{i=1}^{N} F_i(\omega)F_i^*(\omega)\right\rangle,\end{aligned}$$

where $F_i(\omega) = \int_{-\infty}^{\infty} s_i(t)e^{-j\omega t}dt$.

Since the expected value is a linear operator, we can write

$$\begin{aligned}\langle P(\omega)\rangle &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \left\langle F_i(\omega)F_j^*(\omega)\right\rangle + \sum_{i=1}^{N}\left\langle F_i(\omega)F_i^*(\omega)\right\rangle = \\ &= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \left\langle |F_i(\omega)||F_j^*(\omega)| \cos\left[\theta_i(\omega) - \theta_j(\omega)\right]\right\rangle + \\ &+ \sum_{i=1}^{N}\left\langle F_i(\omega)F_i^*(\omega)\right\rangle,\end{aligned} \quad (7)$$

where $\theta_i(\omega) = p_i(\omega)\left(\phi_{s_i} + \pi/2\right)$ is the Fourier phase (see Eq. (5)) and the Bessel line index $p_i(\omega)$ is given by

$$p_i(\omega) = \frac{\omega - n\omega_0}{\omega_{s_i}}.$$

For values of ω such that $p_i(\omega)$ is not an integer we have $F_i(\omega) = 0$.

The assumption of purely incoherent synchrotron motion ensures that ϕ_{s_i} is uniformly distributed over the unit circle. This distribution has several interesting properties [10], among which we will mention two. Let X be a random variable uniformly distributed over the unit circle. Firstly, for any non-zero integer k and arbitrary constant α, we have that $kX + \alpha$ is also uniformly distributed over the unit circle. This implies that $\theta_i(\omega)$ is uniformly distributed for all $p_i(\omega) \neq 0$. Secondly, let Y be any random variable on the unit circle. Then, $X + Y$ is uniformly distributed over the unit circle. Consequently $\theta_i(\omega) - \theta_j(\omega)$ is uniformly distributed. Moreover, values of $\theta_i(\omega)$ are independent from those of $|F_i(\omega)|$. This fact needs clarification, as both depend on ω via $p_i(\omega)$. Nevertheless, as discussed before, $p_i(\omega)$ takes only integer values and therefore has no influence on the probability distribution of $\theta_i(\omega)$. As a consequence, $\theta_i(\omega)$ and $|F_i(\omega)|$ are independent. We then have

ISBN: 978-3-95450-204-2 ISSN: 2673-5350 doi:10.18429/JACoW-IBIC2019-MOCO03

$$\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\langle |F_i(\omega)||F_j^*(\omega)|\cos\left[\theta_i(\omega)-\theta_j(\omega)\right]\right\rangle =$$

$$= \sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left\langle |F_i||F_j^*(\omega)|\right\rangle \underbrace{\left\langle \cos\left[\theta_i(\omega)-\theta_j(\omega)\right]\right\rangle}_{0} = 0,$$

as $\theta_i(\omega) - \theta_j(\omega)$ is uniformly distributed and $\int_0^{2\pi}\cos(x)dx = 0$. We can conclude that

$$\langle P(\omega)\rangle = \sum_{i=1}^{N}\left\langle F_i(\omega)F_i^*(\omega)\right\rangle, \tag{8}$$

so the expected PSD is just the sum of single particle contributions.

MATRIX FORMALISM

Based on what was explained in the previous section, we may assume, that the time averaged cumulative power spectrum of N particles is equal to the sum of the individual particle spectra. From Eq. (5) we know, that differences in the individual particle spectra depend only on the particle's synchrotron amplitude, as synchrotron frequency can be expressed as a function of the amplitude (Eq. (2)). We do not take synchrotron phase into consideration, as phase will not influence a single particle's spectrum. Therefore the Schottky spectra are explicitly related to the distribution of synchrotron amplitudes.

Let us assume that we know the distribution $g(\widehat{\tau})$ of synchrotron amplitudes amongst the particles. We may then calculate $\langle P(\omega)\rangle$, the power at a given frequency, as:

$$\langle P(\omega)\rangle = \int_0^{\infty} g(\widehat{\tau})P(\omega,\widehat{\tau})d\widehat{\tau}, \tag{9}$$

where $P(\omega,\widehat{\tau})$ is the PSD at frequency ω of a particle with synchrotron amplitude $\widehat{\tau}$. This should be seen as the continuous analogue of Eq. (8). If we discretise $g(\widehat{\tau})$, then Eq. (9) takes the form:

$$\langle P(\omega)\rangle = \sum_{i=0}^{\infty} g(\widehat{\tau}_i)P(\omega,\widehat{\tau}_i).$$

The above equation can be expressed in terms of discrete frequencies, written in matrix form as:

$$\underbrace{\begin{bmatrix} P(\omega_1,\widehat{\tau}_1) & \cdots & P(\omega_1,\widehat{\tau}_n) \\ P(\omega_2,\widehat{\tau}_1) & \cdots & P(\omega_2,\widehat{\tau}_n) \\ \vdots & \ddots & \vdots \\ P(\omega_m,\widehat{\tau}_1) & \cdots & P(\omega_m,\widehat{\tau}_n) \end{bmatrix}}_{\mathcal{M}} \cdot \underbrace{\begin{bmatrix} g(\widehat{\tau}_1) \\ g(\widehat{\tau}_2) \\ \vdots \\ g(\widehat{\tau}_n) \end{bmatrix}}_{\mathcal{A}} = \underbrace{\begin{bmatrix} \langle P(\omega_1)\rangle \\ \langle P(\omega_2)\rangle \\ \vdots \\ \langle P(\omega_m)\rangle \end{bmatrix}}_{\mathcal{S}}. \tag{10}$$

We can see, that the columns of matrix $\mathcal{M}$ correspond to the spectrum of a single particle with synchrotron amplitude $\widehat{\tau}_i$. Vector $\mathcal{A}$ represents the synchrotron amplitude density and vector $\mathcal{S}$ is the expected value of the cumulative power density spectrum of many particles, which can be compared with the experimentally obtained Schottky spectrum.

One should note, that as we have

$$P\left(\omega_i,\widehat{\tau}_j\right) = \left|\int_{-\infty}^{\infty} s_j(t)e^{-j\omega_i t}dt\right|^2,$$

where $s_j(t)$ is given by Eq. (5), matrix $\mathcal{M}$ depends on the nominal synchrotron frequency. Therefore, we shall use the notation $\mathcal{M}\left(\omega_{s_0}\right)$.

BUNCH SHAPE CALCULATIONS

Our procedure will be as follows. Let us assume that we are given an averaged experimental Schottky spectrum $\mathcal{S}_{exp}$. Firstly, we need to exclude the frequency bins corresponding to the $p = 0$ satellite, as these add up coherently and Eq. (9) does not hold. We also remove corresponding rows from matrix $\mathcal{M}$. Then, we try to minimize the cost function

$$C\left(\omega_{s_0},\mathcal{A}\right) = \left|\log\left[\mathcal{M}\left(\omega_{s_0}\right)\cdot\mathcal{A}\right] - \log\left[\mathcal{S}_{exp}\right]\right|^2, \tag{11}$$

where the log functions are taken point-wise and $|\cdot|$ is the standard euclidean metric. Having found the optimal ω_{s_0} and $\mathcal{A}$ we can restore the longitudinal bunch shape and synchrotron frequency distribution using Eqs. (2) and (4) respectively.

It is certain that the experimental spectrum is susceptible to noise and finite time of averaging effects. It may therefore happen, that the pair $(\omega_{s_0},\mathcal{A})$ which minimizes $C\left(\omega_{s_0},\mathcal{A}\right)$ is different from the true nominal synchrotron frequency and amplitude density. We have an example of such a situation in Fig. 2.

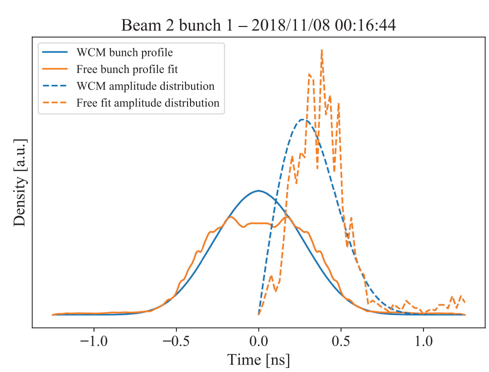

Figure 2: WCM bunch profile and results of bunch profile fitting. Without putting any constraints on the synchrotron amplitude distribution we end up with an exotic bunch shape.

In order to overcome this feature, our proposed solution is based on the assumption that synchrotron amplitude densities follow a Rice distribution [11], which is the distribution of distances from zero for samples taken from a 2D

ISBN: 978-3-95450-204-2 ISSN: 2673-5350 doi:10.18429/JACoW-IBIC2019-MOCO03

univariate normal distribution. It is determined by two parameters, the Gaussian variance σ and the modulus of its mean μ. We base our assumption on observations of bunch shapes measured by the Wall Current Monitor (WCM) [12] and computed synchrotron amplitude densities (via Eq. (4)), which confirm this hypothesis. We present typical amplitude densities together with the corresponding Rice distributions in Fig. 3. Our assumption may be seen as a regularization, that is, introducing information which helps to solve an ill-posed problem by preventing solutions from wrongly compensating the errors.

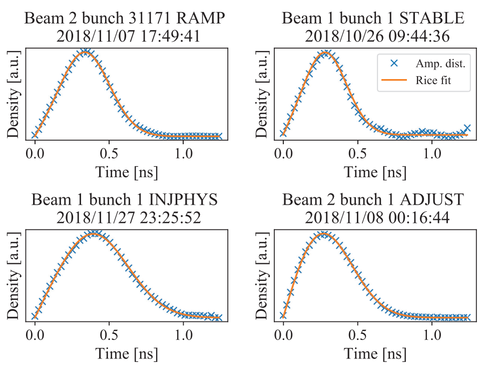

Figure 3: Synchrotron amplitude distributions calculated for different beam modes are shown to follow a Rice distribution. Optimal Rice parameters were found as a result of curve fitting.

Finding a solution to nonlinear problems is not always possible analytically. Therefore we decided to apply a differential evolution algorithm [13] implemented in a SciPy library [14] in order to find parameters which minimize the cost function $C\left(\omega_{s_0}, \mathcal{A}\right)$.

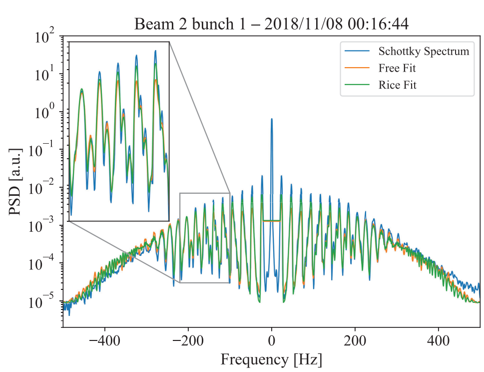

Figure 4: Rice and free fit spectra compared with an experimental Schottky spectrum. There is no fit for the $p = 0$ satellite, as it adds up coherently and Eq. (9) does not hold.

Matrices $\mathcal{M}\left(\omega_{s_0}\right)$ were pre-calculated in order to reduce the computation time needed for evaluation of the cost function. In order to determine the bunch shape and synchrotron frequency distribution, we fit 5 parameters in total. The first three have already been mentioned, these are σ and μ of the Rice distribution and the nominal synchrotron frequency. Additionally, we need to fit the scale, as the magnitude of $\mathcal{S}_{exp}$ may change, and finally, we take into consideration that some information may be masked by noise and we may actually only see the top part of the spectrum. However, this was not found to be the case in the spectra that we have analysed, as the noise parameter was observed to be negligible. Comparing the calculated profiles with those obtained with the WCM, confirms the accuracy of the proposed method. This is shown in Fig. 5.

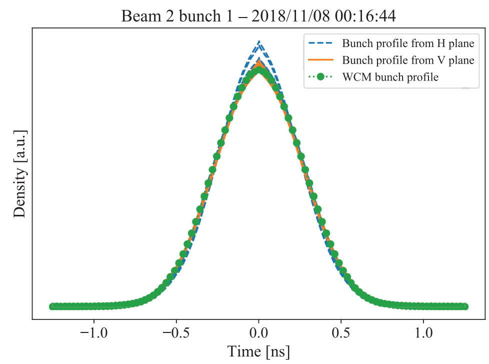

Figure 5: Bunch shapes derived from the Horizontal and Vertical LHC Schottky spectra in a time interval of 100 seconds around WCM measurement.

CONCLUSION

The results obtained have been verified in three stages. Firstly, the experimental Schottky spectrum was compared to one obtained from the optimization procedure, with the conclusion that a Rice-based fit compares well with $\mathcal{S}_{exp}$ (and is similar to the free fit). Secondly, bunch profiles calculated from vertical and horizontal monitors are self-consistent. Finally, comparing the calculated profiles with those obtained with the WCM confirms the accuracy of the proposed method.

The aim of this study was not to transform the Schottky system into a longitudinal profile measurement device, for which the WCMs provide a more direct tool, but to be a step towards improving Schottky-based diagnostics in the LHC. The proposed procedure will now be adapted to transverse signals, which contain additional information on tune and chromaticity, with the measured bunch profiles used as a quality indicator for these derived quantities.

ACKNOWLEDGEMENTS

The authors would like to acknowledge the stimulating discussions with O. R. Jones and T. Lefèvre, as well as the discussions and assistance of T. Argyropoulos, T. Levens, O. Marqversen and M. Wendt.

REFERENCES

[1] R. Pasquinelli and A. Jansson, "Microwave schottky diagnostic systems for the fermilab tevatron, recycler, and cern large hadron collider," *Physical Review Special*

ISBN: 978-3-95450-204-2 ISSN: 2673-5350 doi:10.18429/JACoW-IBIC2019-MOCO03

Topics - Accelerators and Beams, vol. 14, no. 7, 2011. doi:10.1103/PhysRevSTAB.14.072803

[2] M. Betz, O. Jones, T. Lefevre, and M. Wendt, "Bunched-beam schottky monitoring in the lhc," *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 874, pp. 113 – 126, 2017. doi:10.1016/j.nima.2017.08.045

[3] T. Tydecks, D. Alves, T. Levens, M. Wendt, and J. Wenninger, "Status of the LHC Schottky Monitors," in *Proc. 9th International Particle Accelerator Conference (IPAC'18), Vancouver, BC, Canada, April 29-May 4, 2018*, ser. International Particle Accelerator Conference, no. 9. Geneva, Switzerland: JACoW Publishing, June 2018, paper MOPMF058, pp. 247–249. doi:10.18429/JACoW-IPAC2018-MOPMF058

[4] H. Wiedemann, *Particle accelerator physics; 4th ed.* Berlin: Springer, 2015. doi:10.1007/978-3-319-18317-6

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, ninth dover printing, tenth gpo printing ed. New York: Dover, 1964.

[6] K. Ochs, "A comprehensive analytical solution of the nonlinear pendulum," *European Journal of Physics*, vol. 32, p. 479, 02 2011. doi:10.1088/0143-0807/32/2/019

[7] D. Boussard, "Schottky noise and beam transfer function diagnostics," *CERN Accelerator School : 5th Advanced Accelerator Physics Course*, pp. 749 – 782, 1995. doi:10.5170/CERN-1995-006.749

[8] A. Papoulis and S. U. Pillai, *Probability, Random Variables, and Stochastic Processes*, 4th ed. Boston: McGraw Hill, 2002. http://www.worldcat.org/search?qt=worldcat_org_all&q=0071226613

[9] F. W. Stallmann, "Numerical solution of integral equations," *Numer. Math.*, vol. 15, no. 4, pp. 297–305, Aug. 1970. doi:10.1007/BF02165120

[10] S. Jammalamadaka and A. Sengupta, *Topics in Circular Statistics*, ser. Series on multivariate analysis. World Scientific, 2001. https://books.google.ch/books?id=sKqWMGqQXQkC

[11] A. Abdi, C. Tepedelenlioglu, M. Kaveh, and G. Giannakis, "On the estimation of the k parameter for the rice fading distribution," *IEEE Communications Letters*, vol. 5, no. 3, pp. 92–94, March 2001. doi:10.1109/4234.913150

[12] G. Papotti, F. Follin, T. Bohl, and U. Wehrle, "Longitudinal Beam Measurements at the LHC: The LHC Beam Quality Monitor," *Conf. Proc.*, vol. C110904, pp. 1852–1854, 2011.

[13] R. Storn and K. Price, "Differential evolution - a simple and efficient heuristic for global optimization over continuous spaces," *Journal of Global Optimization*, vol. 11, pp. 341–359, 01 1997. doi:10.1023/A:1008202821328

[14] E. Jones, T. Oliphant, P. Peterson *et al.*, "SciPy: Open source scientific tools for Python," 2001–, [Online; accessed September 7, 2019]. http://www.scipy.org/