NOTES ON STEFFEN PARAMETERS OF EXTENDED FRINGE-FIELD QUADRUPOLES

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Abstract

We consider some theoretical aspects of the Steffen hard-edge model of quadrupoles with extended fringe-fields and discuss possibilities of usage of this model in online beam dynamics applications.

INTRODUCTION AND PRELIMINARIES

Optical linear properties of a quadrupole magnet are completely determined by the quadrupole focusing strength $k_0 \geq 0$ and by the gradient shape function $k(s)$ through the equations of motion

$$d^2 z^{\pm} \,/\, ds^2 \;\pm\; k_0 \, k(s) \, z^{\pm} \;=\; 0, \tag{1}$$

where z^+ and z^- are transverse coordinates in the focusing and defocusing plane, respectively. As concerning the gradient shape function, we assume that this function is nonnegative and its maximum value is equal to one, and it is effectively zero outside the interval $[-l_f \,/\, 2,\; l_f \,/\, 2]$ of the longitudinal s-axis. Besides that, because most often in practice the quadrupole magnet is designed to have a transverse plane of symmetry at $s = 0$, we assume also that $k(s)$ is an even function. An example of the function $k(s)$ for one of the European XFEL quadrupoles can be seen in Fig.1.

Conventional Rectangular Model

Equations (1) are linear and therefore the knowledge of their 2×2 focusing and defocusing transport matrices $M^{\pm}(k_0)$ completely solves the problem of the linear beam transport through the quadrupole. Unfortunately, there are only very few types of the shape functions $k(s)$ which allow to find a closed-form analytical solution for the matrices $M^{\pm}(k_0)$. The simplest such shape function is the rectangular function which is zero outside the interval $[-l_e \,/\, 2,\; l_e \,/\, 2]$ and unity inside it, and the rectangular model approximation is still the most commonly used tool for the description of the optical linear properties of quadrupoles. In this model one defines effective field boundary l_e (often named effective length) as the integral

$$l_e \;=\; \int_{-l_f/2}^{l_f/2} k(s) \, ds \tag{2}$$

and calculates $M^{\pm}(k_0)$ using approximation

$$M^{\pm}(k_0) \approx D[(l_f - l_e)\,/\,2]\, Q^{\pm}(l_e,\, k_0)\, D[(l_f - l_e)\,/\,2], \tag{3}$$

where D is the 2×2 drift matrix, and Q^+ and Q^- are the usual 2×2 focusing and defocusing matrices of the hard-edge quadrupole model, respectively. Note that parameters l_e and k_0 are often named effective parameters.

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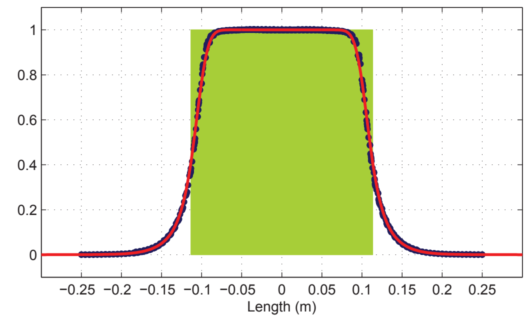

Figure 1: Quadrupole XQI. Red curve shows its gradient shape function $k(s)$, which was obtained (in the form of some combination of the Enge functions) as a fit to the measured data represented in this figure by the blue dots. Green rectangle shows effective field boundary l_e.

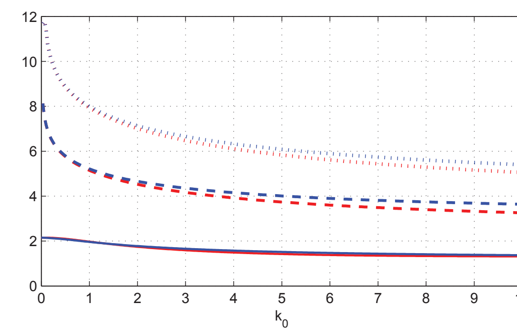

Figure 2: Quadrupole XQI. The number of correct decimal digits in the approximation of the matrices $M^{\pm}(k_0)$ by conventional rectangular model (solid curves), and by zeroth (dashed curves) and first (dotted curves) order Steffen models. Red and blue colors refer to the focusing and defocusing matrices, respectively.

It is believed that approximation (3) is quite good as long as the ratio between the bore radius and the quadrupole length is small enough, and loses its accuracy as this ratio increases. Nevertheless, because one can not rely on such property as "being small enough", the actual precision has to be carefully checked for each specific case. For example, for all European XFEL quadrupoles conventional rectangular model can not provide more than two correct significant decimal digits in the elements of the matrices $M^{\pm}(k_0)$, which does not seem very satisfactory (see Fig.2).

There are many ways to obtain matrices $M^{\pm}(k_0)$ with better accuracy than that of the conventional model, but, unfortunately, most of them also increase the computational time, which, in the next turn, could have negative impact on a number of tasks connected, for example, with large scale optimization of accelerator lattices involving many free parameters or with some real-time online beam dynamics applications. So, in this paper we employ for solution of this problem the Steffen hard-edge model [1], which, after some preliminary work, has practically the same level of the computational complexity as the conventional rectangular model. We would note that although the Steffen approach is not universally known yet, its popularity seems to grow with time (see, for example, [2–6] and references therein).

Steffen Hard-Edge Model

Because the function $k(s)$ is even, each of matrices $M^{\pm}(k_0)$ has only two independent elements, and thus one may hope to find a rectangular type model with different effective parameters for the focusing and defocusing planes whose transfer matrices are exactly equal to the matrices $M^{\pm}(k_0)$. That is actually a basis of the Steffen approach, and as the Steffen quadrupole parameters we understand the values $l^{\pm}$ and $r^{\pm}$ such that the following equations hold

$$M^{\pm}(k_0) = D[(l_f - l^{\pm})/2]\, Q^{\pm}(l^{\pm}, r^{\pm} k_0)\, D[(l_f - l^{\pm})/2]. \quad (4)$$

We call $l^{\pm}$ Steffen effective lengths and $r^{\pm}$ Steffen gradient reduction coefficients. If for some $k_0 > 0$ the Steffen parameters exist, then they can be found as follows:

$$l^{+} = -\frac{\psi^{+} \sin(\psi^{+})}{m_{21}^{+}}, \quad r^{+} = \frac{1}{k_0}\left(\frac{\psi^{+}}{l^{+}}\right)^2, \quad (5)$$

$$l^{-} = \frac{\psi^{-} \sinh(\psi^{-})}{m_{21}^{-}}, \quad r^{-} = \frac{1}{k_0}\left(\frac{\psi^{-}}{l^{-}}\right)^2, \quad (6)$$

where ψ^{+} and ψ^{-} are the positive solutions of the equations

$$\cos(\psi^{+}) + \tfrac{1}{2}\,\psi^{+} \sin(\psi^{+}) = m_{11}^{+} - \tfrac{1}{2}\, l_f m_{21}^{+}, \quad (7)$$

$$\cosh(\psi^{-}) - \tfrac{1}{2}\,\psi^{-} \sinh(\psi^{-}) = m_{11}^{-} - \tfrac{1}{2}\, l_f m_{21}^{-}, \quad (8)$$

and $m_{ij}^{\pm}$ are the elements of the matrices $M^{\pm}(k_0)$.

One sees that if the Steffen parameters exist and are known as functions of k_0, then the calculation of the matrices $M^{\pm}(k_0)$ has the same level of complexity as the calculations made according to the formula (3). The functional dependence $l^{\pm} = l^{\pm}(k_0)$ and $r^{\pm} = r^{\pm}(k_0)$ has to be found only once, beforehand, and then can be kept in some appropriate form as a quadrupole property. For example, for the European XFEL quadrupoles, we found it sufficient to use for this purpose low (up to fifth) order polynomials.

EXISTENCE AND SERIES EXPANSION OF THE STEFFEN PARAMETERS

In this section we prove (due to space limitation, only on the "physical level" of strictness) that for all sufficiently small $k_0 > 0$ there exists a solution for the Steffen parameters which can be expanded into power series of the form

$$l^{\pm} = \sum_{n=0}^{\infty} l_n\, (\pm k_0)^n, \quad r^{\pm} = \sum_{n=0}^{\infty} r_n\, (\pm k_0)^n. \quad (9)$$

Note that our proof is valid only under additional assumption that the function $k(s)$ is non-decreasing on the closed interval $-l_f/2 \le s \le 0$.

Transformation of Steffen Equations

The first step is to expand the right sides of (7) and (8) in powers of k_0. Applying to this problem standard technique of the perturbation theory, one obtains convergent for all values of k_0 series

$$m_{11}^{\pm} - \tfrac{1}{2} l_f m_{21}^{\pm} = 1 + \sum_{n=1}^{\infty} a_n\, (\pm k_0)^n, \quad (10a)$$

$$m_{21}^{\pm} = \sum_{n=1}^{\infty} b_n\, (\pm k_0)^n, \quad (10b)$$

where the coefficients a_n and b_n can be expressed as follows

$$a_n = (-1)^{n+1} \int_{-l_f/2}^{l_f/2} s_1 k(s_1) \int_{-l_f/2}^{s_1} (s_1 - s_2) k(s_2) \cdots \int_{-l_f/2}^{s_{n-1}} (s_{n-1} - s_n)\, k(s_n)\, ds_n \ldots ds_1, \quad (11a)$$

$$b_n = (-1)^{n} \int_{-l_f/2}^{l_f/2} k(s_1) \int_{-l_f/2}^{s_1} (s_1 - s_2) k(s_2) \cdots \int_{-l_f/2}^{s_{n-1}} (s_{n-1} - s_n)\, k(s_n)\, ds_n \ldots ds_1. \quad (11b)$$

As the second step, let us expand also the left sides of (7) and (8) in Taylor series with respect to the variables $\psi^{\pm}$ about the points $\psi^{\pm} = 0$. These series again converge for all values of their expansion arguments and together with (10) allow to combine (7) and (8) into a single equation

$$1 - \sum_{n=1}^{\infty} \frac{n-1}{(2n)!} \left[\mp (\psi^{\pm})^2\right]^n = 1 + \sum_{n=1}^{\infty} a_n (\pm k_0)^n. \quad (12)$$

Comparing the left and right sides of (12) one finds that this equation is solvable for all sufficiently small k_0 (and $\psi^{\pm}$) values if and only if the first nonzero coefficient a_m is negative and its index m is an even number, which is indeed the case because according to (11)

$$a_1 = 0, \quad a_2 = -\frac{l_e}{2} \int_{-l_f/2}^{l_f/2} u^2\, k(u)\, du < 0. \quad (13)$$

Using now expressions for $r^{\pm}$ from (5) and (6), and substituting (9) into them, one obtains

$$(\psi^{\pm})^2 = k_0\, r^{\pm}\, (l^{\pm})^2 = k_0 \sum_{n=0}^{\infty} \omega_n\, (\pm k_0)^n, \quad (14)$$

where

$$\omega_n = \sum_{m=0}^{n} r_m \sum_{p=0}^{n-m} l_p\, l_{n-m-p}. \quad (15)$$

Plugging then (14) into (12), we reduce finally (7) and (8) to the following desired form

$$\sum_{n=2}^{\infty} \left\{ (-1)^n \frac{n-1}{(2n)!} \left[\sum_{m=0}^{\infty} \omega_m (\pm k_0)^m \right]^n + a_n \right\} (\pm k_0)^n = 0. \quad (16)$$

By analogy, the first equations from (5) and (6) can also be brought to the single equation

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$$\left[\sum_{n=0}^{\infty} l_n (\pm k_0)^n\right] \cdot \left[\sum_{n=1}^{\infty} b_n (\pm k_0)^n\right]$$
$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} \left[\sum_{m=0}^{\infty} \omega_m (\pm k_0)^m\right]^n (\pm k_0)^n = 0. \quad (17)$$

Determination of Expansion Coefficients

Equations (15), (16) and (17) allow to determine successively all the coefficients l_n and r_n. First, taking into account that ω_0 must be non-negative, one finds from (16)

$$\omega_0 = \sqrt{-24 a_2}, \quad \omega_1 = \frac{12}{\omega_0}\left(\frac{\omega_0^3}{360} - a_3\right), \quad (18a)$$

$$\omega_2 = \frac{12}{\omega_0}\left(\frac{\omega_0^2 \omega_1}{120} - \frac{\omega_0^4}{13440} - \frac{\omega_1^2}{24} - a_4\right), \quad \ldots, \quad (18b)$$

and then using (17) one obtains l_n

$$l_0 = -\frac{\omega_0}{b_1}, \quad l_1 = \frac{1}{b_1}\left(\frac{\omega_0^2}{6} - \omega_1 - b_2\, l_0\right), \quad (19a)$$

$$l_2 = \frac{1}{b_1}\left(\frac{\omega_0\, \omega_1}{3} - \frac{\omega_0^3}{120} - \omega_2 - b_3\, l_0 - b_2\, l_1\right), \quad \ldots. \quad (19b)$$

Note that $l_0 > 0$ as it is necessary, because $b_1 = -l_e < 0$.

In the last step, r_n can be obtained from (15) as follows

$$r_0 = \omega_0 \,/\, l_0^2, \quad r_1 = (\omega_1 - 2 l_0 l_1 r_0) \,/\, l_0^2, \quad (20a)$$

$$r_2 = \left[\omega_2 - (2 l_0 l_2 + l_1^2) r_0 - 2 l_0 l_1 r_1\right] / \, l_0^2, \quad \ldots. \quad (20b)$$

Zeroth-Order Coefficients

According to the above considerations, the zeroth-order coefficients l_0 and r_0 are given by the following formulas

$$l_0 = \left(\frac{12}{l_e}\int_{-l_f/2}^{l_f/2} s^2 k(s) ds\right)^{\frac{1}{2}}, \quad r_0 = \frac{l_e}{l_0}, \quad (21)$$

but one still has to show that they are physically meaningful, i.e. one still has to prove that $l_0 \leq l_f$. We actually can prove more and can show that

$$l_e \leq l_0 \leq l_f, \quad (22)$$

and that the both inequalities are strict if $k(s)$ is not an rectangular profile (i.e. if the extended fringe-fields are really presented). It is clear that the inequalities (22) are equivalent to the inequalities

$$l_e^3 \leq l_e l_0^2 \leq l_e l_f^2, \quad (23)$$

and the correctness of the inequalities (23) is derivable trivially from our assumptions about the function $k(s)$ and the following integral representations

$$l_e l_f^2 - l_e l_0^2 = 16 \int_{-l_f/2}^{0} \int_{-l_f/2}^{s_1} \int_{-l_f/2}^{s_2} \Big\{ [k(s_1) - k(s_3)]$$
$$+ [k(s_2) - k(s_3)] \Big\} ds_3 ds_2 ds_1, \quad (24)$$

$$l_e l_0^2 - l_e^3$$
$$= 48 \int_{-l_f/2}^{0} \int_{-l_f/2}^{s_1} \int_{-l_f/2}^{s_2} [1 - k(s_1) k(s_2)]\, k(s_3) ds_3 ds_2 ds_1. \quad (25)$$

APPLICATION STRATEGY

The important question of the usage of the Steffen model in practice is the question of the representation of the Steffen parameters as functions of the quadrupole focusing strength k_0. For example, for achieving high accuracy, one can first calculate Steffen parameters numerically with small step in k_0 value and then use the obtained results later in the form of a lookup table combined with some interpolation procedure. But, as we already mentioned, we found it sufficient for our purposes to approximate them using polynomials

$$l_m^{\pm} = \sum_{n=0}^{m} l_n (\pm k_0)^n, \quad r_m^{\pm} = \sum_{n=0}^{m} r_n (\pm k_0)^n, \quad (26)$$

and we call this approximation as mth-order Steffen model. Regarding coefficients l_n and r_n, let us note that we calculate l_0 and r_0 using equations (21), but for the remaining coefficients we actually do not use equations (19) and (20), and find them by direct interpolation of the numerical solutions of the Steffen equations, which means that, in general, they could be somewhat different from the corresponding coefficients of the series (9).

One sees in Fig.2 what kind of improvement can be achieved in comparison with the conventional rectangular model by employing only the zeroth and first order Steffen models, and if one will use the fifth-order Steffen model, then the number of correct digits will be not smaller than ten for all presented k_0 values. Note that the zeroth-order Steffen model, which already shows sizeable improvement, can be used in any standard beam dynamics program (including, for example, MAD) because it is the same as conventional rectangular model but with modified effective parameters.

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