CHARGED PARTICLE TRANSPORT, GAUSSIAN OPTICS, ERROR PROPAGATION: IT'S ALL THE SAME

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Abstract

We derive a correspondence between the parameters used in Gaussian light beam propagation with wavelength λ, beam size w, and wavefront curvature ρ with the description in terms of emittance and Twiss parameters commonly used in charged particle optics. Furthermore, we discuss the analogy of transporting beams to the propagation of measurement uncertainties.

INTRODUCTION

When teaching charged particle optics [1–3] for accelerators we found that pointing out the correspondence between particle optics and the propagation of measurement uncertainties [4] helped the students to better understand the basic concepts. When, at another time, working on experiments where charged particle beams interact with lasers [5, 6] we found that the concepts used in Gaussian optics [7–9] are very similar but use a different framework of ideas. The unifying concept among light and charged particle optics as well as error propagation is the transport of individual entities: light rays and measurement values; and the transport of intervals of the same entities: beam widths and error bars, respectively. In this report I elaborate the correspondence between the three fields and provide methods to translate one into the others.

OPTICS

In optics the propagation of paraxial rays is described using the so-called ABCD method [7], where the coefficients A, B, C, D are matrix elements in the transfer matrix that propagates rays specified by the radial offset r and angle r' with respect to the optical axis. The ray at the second position (r_2, r_2') is given in terms of of those at the first position (r_1, r_1') by

$$\begin{pmatrix} r_2 \\ r_2' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} r_1 \\ r_1' \end{pmatrix} = R \begin{pmatrix} r_1 \\ r_1' \end{pmatrix} . \qquad (1)$$

which defines the transfer matrix R and specifies the behavior of individual rays.

An ensemble of rays emanating from a diffraction limited source in this context is described by the wave front radius of curvature ρ and the width w of the Gaussian distribution in the coordinates r. Note that w and ρ describe properties of an ensemble of many rays, and not individual rays. Using the convention conventionally used in Gaussian optics [8] we describe the radial intensity distribution $I(r)$ by

$$I(r) = I_0\, e^{-2r^2/w^2} = I_0\, e^{-r^2/2\sigma^2} \qquad (2)$$

where the second equality defines the conventional rms beam size σ. We thus deduce that width of the intensity distribution of a optical ensemble of rays or beam, w, is twice the rms beam size radius σ

$$w = 2\sigma\ . \qquad (3)$$

The variation of the beam width w and the wavefront curvature ρ through a beam line described by the transfer matrix specified through A, B, C, D is given by [7]

$$q_2 = \frac{Aq_1 + B}{Cq_1 + D} \qquad (4)$$

where q_1 is an abstract quantity, given in terms of width w_1 and wavefront curvature ρ_1 through

$$\frac{1}{q_1} = \frac{1}{\rho_1} - \frac{i\lambda}{\pi w_1^2}\ . \qquad (5)$$

Here the subscript labels the location and λ is the wavelength of the radiation. Note that the wavefront curvature ρ_1 is given by the real part of $1/q_1$ and the beam width w_1 by the imaginary part. Historically, q is defined in this way because it propagates in a simple way, Eq. (4), and the matrices describing consecutive optical elements are described by normal matrix products of the respective ABCD matrices.

We can also extract the physical quantities w and ρ from q by introducing the real and imaginary part of q by $q = u + iv$ and solve for w and ρ

$$\rho = \frac{u^2 + v^2}{u} \quad \text{and} \quad w^2 = \frac{\lambda}{\pi} \frac{u^2 + v^2}{v}\ . \qquad (6)$$

These expressions can be used to extract the curvature ρ and width w from q.

CHARGED PARTICLE TRANSPORT

The propagation of individual rays of charged particles is described in the same way as paraxial optical rays in Eq. (1). The ensemble of rays – the beam – is described by the matrix of the second moments of the ensemble [3]. It is conventionally called the sigma- or beam-matrix

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xx'} \\ \sigma_{xx'} & \sigma_{x'x'} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \qquad (7)$$

with $\gamma = (1 + \alpha^2)/\beta$. The definition of γ ensures that the determinant of the matrix with β and α has unit determinant. Note also that the beam size σ is related to the emittance and beta function through the relation

$$\sigma^2 = \sigma_{xx} = \varepsilon\beta\ . \qquad (8)$$

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The previous equation only considers one transverse plane, here the horizontal direction. In general there is a second similar equation for the other plane.

The sigma matrix is propagated through the section of beam line from position labeled 1 to position 2 is described by the matrix R with coefficients A, B, C, D by the following expression [1]

$$\begin{pmatrix} \sigma_{xx}^{(2)} & \sigma_{xx'}^{(2)} \\ \sigma_{xx'}^{(2)} & \sigma_{x'x'}^{(2)} \end{pmatrix} = R \begin{pmatrix} \sigma_{xx}^{(1)} & \sigma_{xx'}^{(1)} \\ \sigma_{xx'}^{(1)} & \sigma_{x'x'}^{(1)} \end{pmatrix} R^T \tag{9}$$

which follows from writing the the moments at the second location in terms of those of the first location.

We now proceed to find a correspondence between $\varepsilon, \beta, \alpha$ and the parameters defining the propagation in light optical systems λ, w, ρ.

CLOSE TO A FOCUS

We first explore the correspondence of the two ways of describing the ensemble of rays near a focus or beam waist. The focus is characterized by a flat wavefront, which implies that the curvature of the wavefront is infinite $\rho = \infty$ and we thus have

$$q_1 = -\frac{\pi w_1^2}{i\lambda} \tag{10}$$

at the waist. The beam parameters a distance L away from the waist can be calculated by transporting with the transfer matrix of a drift space

$$R_d = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} . \tag{11}$$

To find q_2 at the position a distance L away we insert q_1 from eq. 10 into Eq. (4) and use the matrix coefficients from R_d. After a little algebra we find

$$q_2 = L + i\frac{\pi w_1^2}{\lambda} \tag{12}$$

and the beam width at location 2 can be determined by using Eq. (6) with the result $w_2^2 = w_1^2 + (\lambda^2/\pi^2)(L^2/w_1^2)$ or, in terms of the rms beam size σ

$$\sigma_2^2 = \sigma_1^2 + \left(\frac{\lambda}{4\pi}\right)^2 \frac{L^2}{\sigma_1^2} . \tag{13}$$

If we now express the rms width σ through the emittance ε and beta function β by virtue of Eq. (8) we find

$$\beta_2^2 = \beta_1^2 + \left(\frac{\lambda}{4\pi\varepsilon}\right)^2 \frac{L^2}{\beta_1^2} . \tag{14}$$

A similar relation can also be deduced by applying the conventional transfer matrix propagation method in Eq. (9) to a sigma matrix shown in Eq. (7) at the waist which has $\alpha = 0$. In that case we find

$$\beta_2^2 = \beta_1^2 + \frac{L^2}{\beta_1^2} . \tag{15}$$

Comparing with Eq. (14) we find that the expression $\lambda/4\pi\varepsilon$ must be unity. This indicates that one part of the sought correspondence is that the 'equivalent emittance' ε of a diffraction limited light beam is given by

$$\varepsilon = \frac{\lambda}{4\pi} . \tag{16}$$

In passing, we note that M^2, the quantity used to characterize the deviation from a diffraction limited beam in light optics is simply the 'equivalent emittance' in units of the emittance of a diffraction limited light beam.

It is instructive to use Eq. (16) to simplify the definition of q in Eq. (5) by substituting first σ for w and then using Eq. (8) to obtain

$$\frac{1}{q} = \frac{1}{\rho} - \frac{i}{\beta} . \tag{17}$$

We see that the imaginary part of q is directly related to the beta function in a simple way.

GENERAL BEAMS

We start from a general initial beam described by the second part of eq. 7 and use Eq. (9) with the transfer matrix R expressed in terms of A, B, C, D. Explicitely performing the matrix multiplications we find

$$\begin{aligned} \varepsilon_2\beta_2 &= \varepsilon_1\left(A^2\beta_1 - 2AB\alpha_1 + B^2\frac{1+\alpha_1^2}{\beta_1}\right) \qquad (18) \\ \varepsilon_2\alpha_2 &= \varepsilon_1\left(AC\beta_1 - (BC+AD)\alpha_1 + BD\frac{1+\alpha_1^2}{\beta_1}\right) . \end{aligned}$$

Now we can also do the same calculation using the optical q parameters. We start by considering the q parameter at location 1

$$q_1 = \frac{\beta_1\rho_1}{\beta_1 - i\rho_1} \tag{19}$$

which follows directly from Eq. (17). At location 2 the same relation applies with subscript 1 replaced by 2. We can thus write

$$\frac{1}{q_2} = \frac{1}{\rho_2} - \frac{i}{\beta_2} = \frac{Cq_1 + D}{Aq_1 + B} = \frac{C\beta_1\rho_1 + D\beta_1 - iD\rho_1}{A\beta_1\rho_1 + B\beta_1 - iB\rho_1} \tag{20}$$

where we used Eq. (19) to express q_1 in terms of β_1 and ρ_1. The real and imaginary parts of the last expression can be obtained by some lengthy algebra. For the imaginary part that is related to the beta function we find

$$-\beta_2 = \frac{A^2\beta_1^2\rho_1^2 + 2AB\beta_1^2\rho_1 + B^2(\beta_1^2 + \rho_1^2)}{(-AD + BC)\beta_1\rho_1^2} \tag{21}$$

Canceling some terms and a little reordering yields

$$\beta_2 = \frac{1}{AD - BC}\left[A^2\beta_1 + 2AB\frac{\beta_1}{\rho_1} + B^2\left(\frac{1}{\beta_1} + \frac{\beta_1}{\rho_1^2}\right)\right] \tag{22}$$

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This equation we can now compare with Eq. (18) and immediately see the correspondence

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{\lambda_2}{\lambda_1} = \frac{1}{AD - BC} \quad \text{and} \quad \alpha_1 = -\frac{\beta_1}{\rho_1} . \tag{23}$$

Now a little discussion is in order. The term $AD - BC$ is actually the determinant of the transfer matrix that is usually unity in beam transport systems and it follows that the emittances are conserved quantities in such systems. In optical systems the transfer matrices that describe the transition from a medium with one refractive index to another has non-unity determinant. This reflects the fact that the wavelength λ in the two media is different and related by the ratio of the refractive index, a fact that is reflected in the first of Eqs. (23). In beam optical systems a non-unity determinant of a transfer matrix corresponds to damping or anti-damping and that affects the emittance, which is also reflected in Eq. (23). The second equation relates the Twiss parameter α to the wavefront curvature ρ.

We can conclude that there is an equivalence of the description of Gaussian wavefront propagation and the transport method used in charged particle beam propagation. The correspondence is summarized by

$$\varepsilon = \frac{\lambda}{4\pi}, \quad \beta = \frac{\pi w^2}{\lambda} = z_R, \quad \alpha = -\frac{\beta}{\rho} = -\frac{\pi w^2}{\lambda \rho} \tag{24}$$

which can be used to translate optical parameter λ, w, ρ to charged particle optical $\varepsilon, \beta, \alpha$. Note that the right hand side of the middle equation for β is the common definition of the Rayleigh length z_R used in optics and which turns out to be the same as the beta function β used in charged particle optics. Inverting the relations we have

$$\lambda = 4\pi\varepsilon, \qquad w = 2\sqrt{\varepsilon\beta}, \qquad \rho = -\frac{\beta}{\alpha} . \tag{25}$$

Using these relations it is easy to translate between the two descriptions of optical system in terms of q or sigma matrices. The descriptions are equivalent.

The equivalence implies also that the limitations of the concepts are equal. Both are paraxial approximation governed by linear (matrix) equations. Non-linearities in charged particle beams appear as non-Gaussian beams, whereas the light optics they appear as higher order modes in terms of Hermite of Laguerre polynomials.

ERROR PROPAGATION

The measurement uncertainties (or 'error bars') for a number of measured values X_i are conventionally assembled in a covariance matrix $C(X)$, whose diagonal component $C(X)_{ii}$ denote the square of the uncertainty for the corresponding value X_i and the off-diagonal elements $C(X)_{ij}$ with $i \neq j$ denote the correlations among the values X_i and X_j. It is well-known that the covariance matrix $C(Y)$ of a set of new variables Y_k, expressed through the original values by $Y_k(X_l)$, is given by [4]

$$C(Y) = JC(X)J^t \tag{26}$$

where $J(Y, X)_{ij} = \partial Y_i / \partial X_j$ is the Jacobi matrix for the transformation of the variables X to Y and J^t denotes the transpose of the matrix J.

In the case that we have a linear change of variables $Y = RX$ where Y and X are vectors and R the matrix that effects the linear transformation, then R is the Jacobi matrix and Eq. (26) directly corresponds to Eq. (9). The sigma matrix in beam physics thus corresponds to the covariance matrix featuring in the description of measurement errors and the transfer matrices (or ABCD matrices in optics) corresponds to the Jacobi matrix that propagate the variables of phase space at one location in the beam line to those at another.

One might push the analogy even further. The centroid of a charged particle beam corresponds to a measured variable and the beam optical sigma matrix to the correlation matrix of measurement uncertainties. Moreover, the diagonal elements of the correlation matrix correspond to beam sizes or divergencies of the particle or laser beams and the off-diagonal elements to the inverse of the wave front curvature R or the Twiss parameter α, respectively. Propagating the beam to a new location in the beam line with the help of a transfer matrix R is equivalent of propagating the covariance matrix with the Jacobi matrix, which in itself corresponds to the transfer matrix.

CONCLUSIONS

We finally summarize the correspondences of three cases in Table 1.

Table 1: Particle, Light and Error Propagation Correspondence

Particle Optics	Light Optics	Error Propagation
Transfer matrix R	$ABCD-$matrix	Jacobi matrix
Beam matrix σ	-	Covariance matrix
Beam size σ	Width w	Error bar
Twiss β	Rayleigh length z_R	-
Twiss α	Curvature $1/\rho$	correlation
Emittance ε	$\lambda/4\pi$	-

Not all concepts appear in all combinations, but being aware of the correspondences will help understand the underlying mechanisms and also aid communication among colleagues in different fields.

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