DYNAMIC APERTURE CONTROL IN ACCELERATOR LATTICES WITH MULTIPOLE POTENTIALS

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Abstract

We apply two analytical methods to control accelerator dynamic aperture (DA) with multipole potentials. Both methods assume that accelerator model can be represented as a product of unperturbed and perturbed exponential operators with exponent of perturbed operator given as formal power series in perturbation parameter and known dependence of series coefficients on lattice parameters such as multipole strength distribution. Normal form method can be applied to the above representation and then lattice parameters are used to control normal form Hamiltonian coefficients. Hamiltonian control can be used to compute control term and lattice parameters are then fitted to approximate corresponding controlled operator. Theoretical results as well as model examples are presented.

INTRODUCTION

In this paper we try to enlarge DA limited by sextupoles by constructing symbolic objective functions with analytical tools such as normal forms [1,2] and Hamiltonian control theory [3,4]. Analytical computations are generally complex but results depend explicitly on system parameters. Thus analytical results can be used as a complement to numerical DA optimization [5] by providing good initial values for system parameters and additional objective functions.

ACCELERATOR MODEL

Single particle dynamics can be described in terms of compositional operators [1,6]. Evolution of phase space in such system with operator $\mathcal{M}$ is given by:

$$z_\beta = \mathcal{M}\mathcal{I}(z_\alpha)$$

where $\mathcal{I}$ is an identity function, z_α and z_β are initial and final phase space points, respectively. In the case of accelerators it is convenient to express compositional operator as a product of functionally independent parts, e.g. each part describes propagation through a certain accelerator element:

$$\mathcal{M} = \mathcal{M}_1 \mathcal{M}_2 \dots \mathcal{M}_n$$

where $\mathcal{M}_\alpha = \exp([F_\alpha])$ is an element operator and Lie operator is defined as $[F]G := \partial_q F \partial_p G - \partial_p F \partial_q G$. Furthermore, locally autonomous dynamics is assumed and thus generators are of the form:

$$F_\alpha = -s_\alpha H_\alpha = A_\alpha + B_\alpha(\varepsilon)$$

where s_α being element length, H_α being element Hamiltonian and generator can be further splitted into unperturbed part A_α and nonlinear perturbation $B_\alpha(\varepsilon)$.

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Our goal is to factorize $\mathcal{M}$ into a product of unperturbed and perturbed operators. To do so, we first need to decompose individual operators:

$$\mathcal{M}_\alpha = \exp([A_\alpha + B_\alpha(\varepsilon)]) = \exp([A_\alpha])\exp([C_\alpha(\varepsilon)])$$

and find coefficients of $C_\alpha(\varepsilon) = C_\alpha^{(1)}\varepsilon + C_\alpha^{(2)}\varepsilon^2 + \dots$ up to some predefined order ε^n. For multipole perturbation we associate ε^n with a homogeneous polynomial P_{n+2} of degree $n+2$. Decomposition can be motivated by symplectic integrators [7]. In this case element is first splitted into slices and then factorized with BCH formula [8] and operator identities. Another possibility is to use BCH directly:

$$\exp([C_\alpha(\varepsilon)]) := \exp([-A_\alpha])\exp([A_\alpha] + [B_\alpha(\varepsilon)])$$

This can be done for each slice or exact expression can be obtained for pure multipole magnets, i.e. when $A_\alpha = A_\alpha(p)$. Decomposed full operator is of the form:

$$\mathcal{M} = \mathcal{M}[0, 1]\mathcal{M}[1]\mathcal{M}[1, 2] \dots \mathcal{M}[k-1, k]\mathcal{M}[k]\mathcal{M}[k, k+1]$$

where $\mathcal{M}[\alpha] := \exp([C_\alpha(\varepsilon)])$ and doubly indexed operators $\mathcal{M}[\alpha, \beta]$ describe unperturbed motion between perturbations. Using operator identities one gets:

$$\mathcal{M} = \mathcal{M}[0, k+1]\hat{\mathcal{M}}[1] \dots \hat{\mathcal{M}}[k]$$

where $\hat{\mathcal{M}}[\alpha] = \exp([\mathcal{M}^{-1}[\alpha, k+1]C_\alpha(\varepsilon)])$ and the product $\hat{\mathcal{M}}[1] \dots \hat{\mathcal{M}}[k]$ can be then computed leading to the desired factorized form:

$$\mathcal{M} = \mathcal{M}[0, k+1]\exp([X(\varepsilon)]) \tag{1}$$

where $X(\varepsilon) = X^{(1)}\varepsilon + X^{(2)}\varepsilon^2 + \dots$ describes the action of nonlinear perturbation and dependence of it's coefficients on parameters is assumed to be known symbolically.

Another representation can be obtained with Floquet transformation $\mathcal{M}[\alpha, k+1] = \mathcal{A}^{-1}[\alpha]\mathcal{R}[\alpha, k+1]\mathcal{A}[k+1]$ and the operator in this case is:

$$\mathcal{M} = \mathcal{A}^{-1}[0]\mathcal{R}[0, k+1]\exp([Y(\varepsilon)])\mathcal{A}[k+1] \tag{2}$$

Factorized operators given by eq. (1) and (2) can be then used for normal form and Hamiltonian control computations with generic coefficients for observables $X(\varepsilon)$ and $Y(\varepsilon)$.

In this paper we use simple FODO cell with fixed sextupoles as a test problem for DA control. Lattice functions and as well as comparison with analytical computations up to order four are shown in Fig. 1. We also note that nonlinear pseudo–Hamiltonian function $X(\varepsilon)$ computed analytically matches the one obtained with COSY–INFINITY [9].
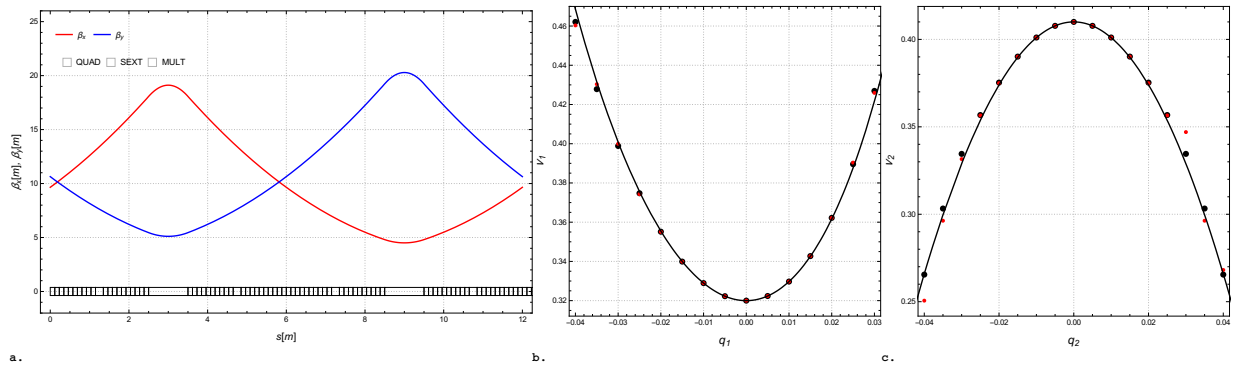
Figure 1: (a.) lattice functions and profile for one cell, full lattice consists of twelve identical cells with $\nu_x = 2.68$ and $\nu_y = 2.41$, (b., c.) horizontal and vertical frequencies as function of corresponding coordinates for element-by-element tracking (black dots), perturbation computed up to order four (red dots) and normal form result (black line).

NORMAL FORM COMPUTATION

The goal of the normal form computation is to find operator $\mathcal{N}$ conjugate to $\mathcal{M}$ that is simpler in some sense:

$$\mathcal{N} = \mathcal{R}\exp([K]) = \exp([T])\mathcal{M}\exp([-T]) \qquad (3)$$

where $\mathcal{M}$ is given by eq. (2), $\mathcal{R} = \exp([-2\pi\nu J])$ is an unperturbed part, $\exp([K])$ is nonlinear part of the normal form and $\exp([T])$ is a normal form transformation. Homological equations for K and T can be deduced from:

$$e^{[K]} = \mathcal{R}^{-1}e^{[T]}\mathcal{R}e^{[Y]}e^{[-T]} = e^{[X]}e^{[Y]}e^{[Z]}$$

by using BCH formula and expanding all functions into formal power series $F = F^{(1)}\varepsilon + F^{(2)}\varepsilon^2 + \ldots$ in ε parameter. Thus homological equations of order k are of the form:

$$K^{(k)} = X^{(k)} + Y^{(k)} + Z^{(k)} + W^{(k)}$$

where $W^{(k)}$ depends on previous orders. To solve these equations all observables are transformed to resonance basis $F^{(k)} = \sum F^{(k)}[n, m]Q[n, m]$ with basis resonance functions defined as $Q[n, m] := J^{n/2}e^{im\varphi}$. Finally, solution for transformation coefficients is:

$$T^{(k)}[n, m] = 1/2\left(1 - i\cot(\pi m\nu)\right)\left(W^{(k)}[n, m] - K^{(k)}[n, m]\right)$$

For the case when $m\nu \neq p \in \mathbb{Z}$ is assumed only terms with $m = 0$ contribute to normal form. Then K depends only on actions and thus commutes with $\mathcal{R}$. Such full normal form Hamiltonian allows one to obtain the dependence of frequencies on amplitude (Fig. 1). Another possibility is to keep strong resonances in normal form even though resonance condition is fulfilled only approximately.

For DA control we can start with full normalization and then fit parameters to reduce frequency dependence on the amplitude and thus reduce tune footprint size. But this procedure can lead to growth of resonance strengths and such resonances (that can be selected based on FMA) should be included into normal form and reduced as well.

HAMILTONIAN CONTROL THEORY

The goal of Hamiltonian control theory is to find a control operator $\exp([K])$ such that $K(\varepsilon) = K^{(2)}\varepsilon^2 + \ldots$ and the controlled operator:

$$\mathcal{M}_K = \mathcal{M}\exp([K]) = \mathcal{R}\exp([Y])\exp([K])$$

is conjugate to a system that is close to unperturbed one:

$$\exp([T])\mathcal{M}_K\exp([-T]) = \mathcal{R}\exp([\mathcal{G}_R Y])$$

This is the case when control operator is defined as:

$$e^{[K]} := e^{[-Y]}e^{[(\mathcal{G}_N - \mathcal{G})Y]}e^{[\mathcal{G}_R Y]}e^{[\mathcal{G}Y]}$$

where $\mathcal{R}$ is an unperturbed part of uncontrolled operator $\mathcal{M}$ given by eq. (2), $\mathcal{G} := \mathcal{G}(1 - \mathcal{R}^{-1})\mathcal{G}$ is a pseudo-inverse operator of $1 - \mathcal{R}^{-1}$, $\mathcal{G}_N := (1 - \mathcal{R}^{-1})\mathcal{G}$ is the non-resonant operator, $\mathcal{G}_R := 1 - \mathcal{G}_N$ is the resonant operator and $T := \mathcal{G}Y$ is the transformation. Controlled operator is then given by:

$$\mathcal{M}_K = \mathcal{R}e^{[-\mathcal{R}^{-1}\mathcal{G}Y]}e^{[\mathcal{G}_R Y]}e^{[\mathcal{G}Y]} = \mathcal{R}e^{[Y_K]} \qquad (4)$$

Perturbation Y is transformed into resonance basis and the action of the above operators on the basis function is:

$$\mathcal{G}Q[n, m] = 1/2(1 - i\cot(\pi m\nu))\Delta(m\nu \notin \mathbb{Z})Q[n, m]$$
$$\mathcal{R}^{-1}\mathcal{G}Q[n, m] = -1/2(1 + i\cot(\pi m\nu))\Delta(m\nu \notin \mathbb{Z})Q[n, m]$$
$$\mathcal{G}_N Q[n, m] = \Delta(m\nu \notin \mathbb{Z})Q[n, m]$$
$$\mathcal{G}_R Q[n, m] = \Delta(m\nu \in \mathbb{Z})Q[n, m]$$

where function Δ is defined as $\Delta(\top) := 1$ and $\Delta(\bot) := 0$.

In general the closed form of controlled operator can't be obtained, but it can be approximated up to some order in perturbation parameter $Y_K(\varepsilon) = Y_K^{(1)}\varepsilon + Y_K^{(2)}\varepsilon^2 + \ldots$ by using BCH formula. Then system parameters should be fitted for control to be realized up to some order. Hamiltonian control doesn't change first order properties of a system, thus to construct controlled operator one can chose only first order perturbation $Y(\varepsilon) = Y^{(1)}\varepsilon$ and, since $Y^{(1)}$ is associated with homogeneous polynomial of degree three, $\mathcal{G}_R Y = 0$. Furthermore, normal form of controlled operator has $K = 0$ and $T = T^{(1)}\varepsilon$. The effect of formal control on DA is shown in Fig. 2.
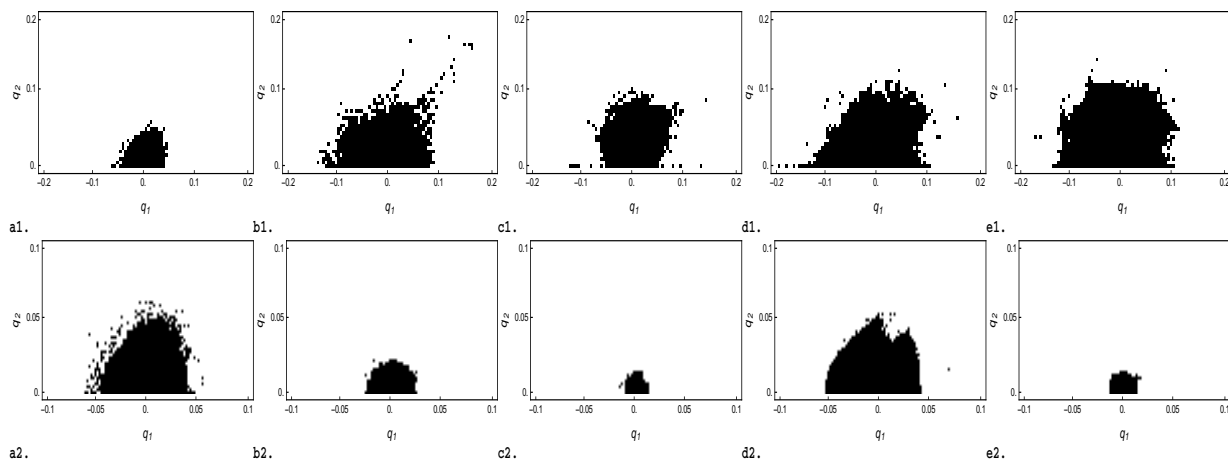
Figure 2: Dynamic aperture of original system (a1., a2.), first order nonlinear perturbation (b1.), second order formal controlled system (c1.), third order formal controlled system (d1.), fourth order formal controlled system (e1.), L2 realization of second order controlled system (b2.), L1 realization of second order controlled system (c2.), L2 realization of zero K_2 normal form (d2.), L1 realization of zero K_2 normal form (e2.).

PARAMETERS FITTING

Realization of normal form and Hamiltonian control objectives can be reduced to minimization problem, since in both cases one needs to fit system parameters (strength of multipoles and their positions) so that coefficients of some polynomial observable have specific values.

If positions of multipoles are allowed to vary, the minimization problem is intrinsically nonlinear. But for fixed multipole positions the problem can be partially reduced to linear minimization. This is possible since leading order dependence of k'th coefficients on k'th order multipole amplitudes is linear. And thus one needs to solve linear problem for each order. Moreover, the problem can be linear for several order at once, e.g. second and third orders are linear with respect to octupole strength for fixed sextupoles.

It should be noted that since fitting is performed up to some order, the effect of higher orders is assumed to be negligible which might not be a good assumption.

RESULTS AND CONCLUSION

We've tried to realise second order controlled system and second order full normal form with zero coefficients by placing octupoles in drift spaces for simple FODO accelerator lattice. The results of DA computation for octupole amplitudes obtained by means of L2 and L1 [10] minimization are shown in Fig. 2. As it can be seen, DA is reduced for most cases except zero K_2 with L2 minimization for which DA is slightly increased. DA is reduced due to large fitted octupole amplitudes that leads to modification of higher order coefficients which results not negligible contribution of higher order terms.

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