

ANALYTICAL STUDIES OF ION BEAM EVOLUTION UNDER COHERENT ELECTRON COOLING*

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Abstract

In the presence of coherent electron cooling (CeC), the evolution of the longitudinal profile of a circulating ion bunch can be described by the 1-D Fokker-Planck equation. We show that, in the absence of diffusion, the 1-D equation can be solved analytically for certain dependence of cooling force on the synchrotron amplitude. For more general cases with finite diffusion, we solved the 1-D Fokker-Planck equation numerically and the numerical solutions have been used to benchmark our simulation code as well as providing fast estimations of the cooling effects.

I. INTRODUCTION

The future electron-ion collider demands a strong hadron cooling technique to reach the luminosity level where all the relevant physics can be fully covered [1]. As a potential candidate to provide such a cooling technique, the concept of Coherent electron Cooling (CeC) will be tested at RHIC in the next year [2]. Both analytical and simulation tools are currently under development to predict the ion bunch evolution in the presence of CeC, which is essential for diagnosing as well as optimizing the cooling system. While simulation through macro-particle tracking is the most straightforward approach in predicting the ion bunch evolution under cooling, analytical tools are needed in both benchmarking the simulation code and providing a fast estimate for the ion bunch profile.

Since the proof of CeC principle experiment involves only longitudinal cooling, the evolution of the ion bunch under cooling can be described by the 1-D Fokker-Planck equation. The analytical tools are being developed to solve the 1-D Fokker-Planck equation with given cooling rate and diffusion coefficient. We review the 1-D Fokker-Planck equation and its equilibrium solutions in section II. In section III, we derived an analytical solution of the 1-D Fokker-Planck equation for a specific functional form of the cooling rate, in the limit of vanishing diffusion coefficient. Section IV contains the numerical solution of the 1-D Fokker-Planck equation for finite diffusion coefficient and arbitrary functional form of the cooling rate. It is shown that the numerical solution agrees with the analytical solution at the proper limits taken by the latter. We summarize in section V.

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II. 1-D FOKKER-PLANCK EQUATION

In the cooling section of a CeC system, a circulating ion sees a coherent energy kick induced by itself to correct its energy error as well as a random energy kick induced by its neighbours (electrons and ions). In addition the circulating ions also get random kicks from the Intra-beam scattering (IBS). In the presence of both the coherent cooling forces and the random diffusive kicks, the evolution of the longitudinal phase space density is described by the 1-D Fokker-Planck equation [3]:

$$\frac{\partial}{\partial t} F(I, t) - \frac{\partial}{\partial I} (\zeta(I) \cdot I \cdot F(I, t)) - \frac{\partial}{\partial I} \left(I \cdot D(I) \cdot \frac{\partial F(I, t)}{\partial I} \right) = 0, \quad (1)$$

where $F(I, t)$ is the longitudinal phase space density averaged over one synchrotron oscillation, I is the amplitude of synchrotron oscillation, $\zeta(I)$ and $D(I)$ are the cooling rate and the diffusion coefficient averaged over one synchrotron oscillation. The diffusion coefficients are to be calculated from the summation of all random kicks.

At equilibrium, the first term at the L.H.S. of eq. (1) vanishes and we obtain the following solution for the equilibrium distribution of the ion beam:

$$F_{eq}(I) = A \exp \left\{ - \int \frac{\zeta(I)}{D(I)} dI \right\}. \quad (2)$$

III. ANALYTICAL SOLUTIONS IN THE LIMIT OF ZERO DIFFUSION

In the limit of $D(I) = 0$, eq. (1) becomes

$$\frac{\partial}{\partial t} F(I, t) - \frac{\partial}{\partial I} (\zeta(I) \cdot I \cdot F(I, t)) = 0. \quad (3)$$

With the following definition:

$$\tilde{F}(I, t) \equiv \zeta(I) \cdot I \cdot F(I, t), \quad (4)$$

and

$$\alpha(I) \equiv I \cdot \zeta(I), \quad (5)$$

eq. (3) can be rewritten as

$$\frac{\partial}{\partial t} \tilde{F}(I, t) - \alpha(I) \frac{\partial}{\partial I} \tilde{F}(I, t) = 0. \quad (6)$$

If the coefficient $\alpha(I)$ does not depend on I , i.e.

$\alpha(I) = \alpha_0$, eq. (6) reduces to the wave equation

$$\frac{\partial}{\partial t} \tilde{F}_w(I, t) - \alpha_0 \frac{\partial}{\partial I} \tilde{F}_w(I, t) = 0, \quad (7)$$

which has the general solution of the form

$$\tilde{F}_w(I, t) = G_w(I + \alpha_0 t). \quad (8)$$

Inspired by above observation, we assume the solution of eq. (6) satisfies:

$$\tilde{F}(X(\tau), Y(\tau)) = C_1, \quad (9)$$

where

$$\begin{cases} I = X(\tau) \\ t = Y(\tau) \end{cases} \quad (10)$$

defines a contour in the (I, t) plane, τ is a parameter determines the specific location along the contour and C_1 is the value that \tilde{F} takes at the contour that does not depend on the specific location parameter τ . Taking the first derivative of eq. (9) with respect to τ yields

$$\begin{aligned} & \frac{d}{d\tau} \tilde{F}(X(\tau), Y(\tau)) \\ &= \frac{dX(\tau)}{d\tau} \cdot \frac{\partial \tilde{F}(I, t)}{\partial I} \Big|_{I=X(\tau)} + \frac{dY(\tau)}{d\tau} \cdot \frac{\partial \tilde{F}(I, t)}{\partial t} \Big|_{I=X(\tau)} \\ &= 0 \end{aligned} \quad (11)$$

Comparing eq. (11) with eq. (6) yields

$$\frac{dX(\tau)}{d\tau} = -\alpha(X(\tau)), \quad (12)$$

and

$$\frac{dY(\tau)}{d\tau} = 0. \quad (13)$$

Solving eq. (13) generates

$$t = \tau + t_0, \quad (14)$$

and the solution of eq. (12) is

$$\tau = -\int \frac{dI}{\alpha(I)} + C, \quad (15)$$

where eq. (10) is applied to eq. (14) and (15). Inserting eq. (15) into eq. (14) leads to

$$C = t - t_0 + \int \frac{dI}{\alpha(I)}. \quad (16)$$

Eq. (16) defines a series of contours and each contour is specified by the value of C . As long as (I, t) stays in the same contour, the value of $\tilde{F}(I, t)$ does not change. Thus, we obtain the general solution of eq. (6):

$$\tilde{F}(I, t) = G(C), \quad (17)$$

with C given by eq. (16). At $t = t_0$, the solution has to satisfy some given initial condition

$$\tilde{F}(I, t_0) = \tilde{F}_0(I), \quad (18)$$

and imposing the initial condition to eq. (17) leads to

$$G\left(\int \frac{dI}{\alpha(I)}\right) = \tilde{F}_0(I). \quad (19)$$

If we define

$$h(I) = \int \frac{dI}{\alpha(I)}, \quad (20)$$

eq. (19) becomes

$$G(h(I)) = \tilde{F}_0(I), \quad (21)$$

and for any value of

$$C = h(I), \quad (22)$$

eq. (21) requires

$$G(C) = \tilde{F}_0(h^{-1}(C)). \quad (23)$$

Inserting eq. (17) and (23) into eq. (4) yields the solution of eq. (3) which satisfy the initial condition eq. (18):

$$F(I, t) = \frac{h^{-1}(C) \zeta(h^{-1}(C)) F_0(h^{-1}(C))}{\zeta(I) \cdot I}. \quad (24)$$

For simplicity, we assume the dependence of the cooling rate on the synchrotron oscillation action takes the following form

$$\zeta(I) = \zeta_0 \frac{I_e}{I + I_e}, \quad (25)$$

where ζ_0 is the cooling rate for ions with zero synchrotron oscillation amplitude (or the reference ion) and I_e is a parameter determining effective cooling range in the longitudinal direction (for CeC PoP, the electron bunch is more than two orders of magnitude shorter than the ion bunch and is determined by the electron bunch length.). Inserting eq. (25) and (5) into eq. (20) yields

$$h(I) = \frac{1}{\zeta_0} \int \frac{1 + \frac{I_e}{I}}{I} dI = \frac{1}{\zeta_0} \ln \left(\frac{I}{I_e} \exp \left(\frac{I}{I_e} \right) \right), \quad (26)$$

and the inverse function of $h(I)$ is given by solving I for equation

$$h(I) = \frac{1}{\zeta_0} \ln \left(\frac{I}{I_e} \exp \left(\frac{I}{I_e} \right) \right) = C. \quad (27)$$

The solution of eq. (27) reads

$$I = h^{-1}(C) = I_e P_{\log} \left[\exp(\zeta_0 C) \right], \quad (28)$$

where the product logarithm function, $P_{\log}(x)$ is the inverse function of

$$w(x) = x e^x, \quad (29)$$

i.e.

$$P_{\log}(x) = w^{-1}(x). \quad (30)$$

Inserting eq.(28) into eq. (24) generates

$$F(I, t) = \left(1 + \frac{I_e}{I} \right) \cdot \frac{P_{\log} \left[\exp(\zeta_0 C) \right] F_0 \left(I_e P_{\log} \left[\exp(\zeta_0 C) \right] \right)}{1 + P_{\log} \left[\exp(\zeta_0 C) \right]}. \quad (31)$$

As an example, we take the initial ion distribution as

$$F_0(I) = \exp \left(-\frac{I}{I_{ion}} \right), \quad (32)$$

where I_{ion} is a parameter determined by the longitudinal emittance of the ion bunch. Inserting eq. (26) and (20) into eq. (16), and taking $t_0 = 0$ lead to

$$C = t + \frac{1}{\zeta_0} \ln \left(\frac{I}{I_e} \exp \left(\frac{I}{I_e} \right) \right). \quad (33)$$

Making use of eq. (32) and (33), eq. (31) becomes

$$F(I, t) = \left(1 + \frac{I_e}{I} \right) \times \frac{P_{\log} \left(\frac{I}{I_e} \exp \left(\zeta_0 t + \frac{I}{I_e} \right) \right) \exp \left(\frac{-I_e}{I_{ion}} P_{\log} \left(\frac{I}{I_e} \exp \left(\zeta_0 t + \frac{I}{I_e} \right) \right) \right)}{1 + P_{\log} \left(\frac{I}{I_e} \exp \left(\zeta_0 t + \frac{I}{I_e} \right) \right)}. \quad (34)$$

The line number density profile of the ion beam is then given by the following expression

$$\rho_{ion}(t, z) = \int_{-\infty}^{\infty} F(z^2 + \delta^2, t) d\delta, \quad (35)$$

where z is the longitudinal location along the ion bunch and δ is the relative energy deviation from the reference ion.

IV. NUMERICAL SOLUTIONS FOR FINITE DIFFUSION

In the presence of non-zero diffusion coefficient, finding analytical solution of eq. (3) is usually difficult and numerical approach is pursued. Using the following definitions of normalized variables: $\bar{I} \equiv I/I_e$, $\bar{t} = t\zeta_0$, $\zeta(I) = \zeta_0 \bar{\zeta}(I)$, $D(I) = D_0 \bar{D}(I)$, $\bar{F}(\bar{I}, \bar{t}) \equiv F(\bar{I} \cdot I_e, \bar{t} \zeta_0)$ and $\bar{r} \equiv \sqrt{\bar{I}}$, eq. (1) can be re-written as

$$\bar{r} \frac{\partial \bar{F}(\bar{r}, \bar{t})}{\partial \bar{t}} + \alpha(\bar{r}) \frac{\partial \bar{F}(\bar{r}, \bar{t})}{\partial \bar{r}} + \beta(\bar{r}) \frac{\partial^2 \bar{F}(\bar{r}, \bar{t})}{\partial \bar{r}^2} + \gamma(\bar{r}) \bar{F}(\bar{r}, \bar{t}) = 0 \quad (36)$$

where

$$\alpha(\bar{r}) = - \left[\frac{\bar{r}^2}{2} \bar{\zeta}(\bar{r}) + \frac{\bar{D}_0}{4} \bar{D}(\bar{r}) + \frac{\bar{D}_0 \bar{r}}{4} \cdot \frac{d\bar{D}(\bar{r})}{d\bar{r}} \right], \quad (37)$$

$$\beta(\bar{r}) = - \frac{\bar{D}_0 \bar{r}}{4} \cdot \bar{D}(\bar{r}), \quad (38)$$

and

$$\gamma(\bar{r}) = - \bar{r} \left[\frac{\bar{r}}{2} \frac{d\bar{\zeta}(\bar{r})}{d\bar{r}} + \bar{\zeta}(\bar{r}) \right]. \quad (39)$$

The difference equation derived from eq. (36) reads

$$\frac{\beta_j}{\Delta \bar{r}^2} \bar{F}_{j-1}^{n+1} + \left(\frac{\bar{r}_j}{\Delta \bar{t}} - \frac{\alpha_j}{\Delta \bar{r}} - 2 \frac{\beta_j}{\Delta \bar{r}^2} + \gamma_j \right) \bar{F}_j^{n+1} + \left(\frac{\alpha_j}{\Delta \bar{r}} + \frac{\beta_j}{\Delta \bar{r}^2} \right) \bar{F}_{j+1}^{n+1} = \frac{\bar{r}_j}{\Delta \bar{t}} \bar{F}_j^n \quad (40)$$

for $2 \leq j < N$ with N being the index of the last bin in the grid of \bar{r} ,

$$- \frac{\alpha_j}{\Delta \bar{r}} \bar{F}_1^{n+1} + \frac{\alpha_j}{\Delta \bar{r}} \bar{F}_2^{n+1} = 0 \quad (41)$$

for $j=1$, and

$$\frac{\beta_N}{\Delta \bar{r}^2} \bar{F}_{N-1}^{n+1} + \left(\gamma_N + \frac{\bar{r}_N}{\Delta \bar{t}} - \frac{\alpha_N}{\Delta \bar{r}} - 2 \frac{\beta_N}{\Delta \bar{r}^2} \right) \bar{F}_N^{n+1} = \frac{\bar{r}_N}{\Delta \bar{t}} \bar{F}_N^n \quad (42)$$

for $j=N$. In deriving eq. (41) and (42), the following boundary conditions are applied:

$$\left. \frac{\partial \bar{F}(\bar{r}, \bar{t})}{\partial \bar{r}} \right|_{\bar{r}=0} = 0, \quad (43)$$

and

$$\bar{F}(\bar{r}, \bar{t}) \Big|_{\bar{r} \rightarrow \infty} = 0. \quad (44)$$

Numerical solution of eq. (40)-(42) is obtained by applying the subroutine, TRIDAG, from Numerical Recipes. After obtaining the phase space density, $\bar{F}(\bar{r}, \bar{t})$, the line number density of the ion bunch is given by

$$\bar{\rho}(\bar{z}, \bar{t}) = \int_{-\infty}^{\infty} \bar{F}(\sqrt{\bar{z}^2 + \bar{\delta}^2}, \bar{t}) d\bar{\delta}. \quad (45)$$

Figure 1 shows the results for ion bunch line density profile after $3\zeta_0^{-1}$ of cooling with the initial ion distribution given by eq. (32), or explicitly $\bar{F}_0(\bar{r}) = \exp(-\bar{r}^2/\bar{r}_0^2)$ and $\bar{r}_0 = \sqrt{I_0/I_e} = 200$. The cooling

rate profile of Fig. 1 is taken from eq. (25), or explicitly

$$\bar{\zeta}(\bar{r}) = \frac{1}{1 + \bar{r}^2}. \quad (46)$$

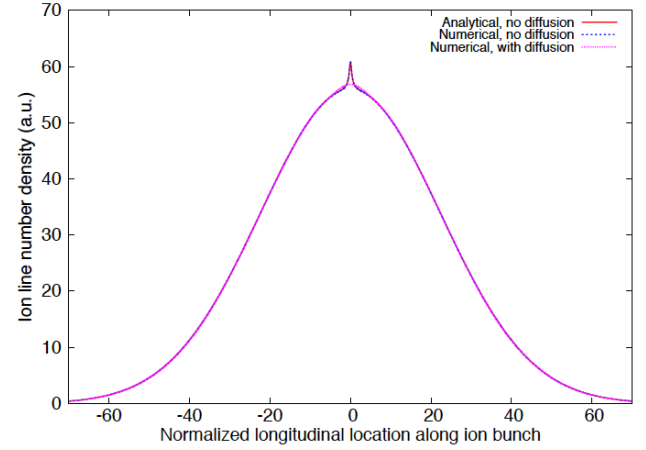


Figure 1: Comparing the numerical solution of eq. (40)-(42) (blue) with the analytical solution as calculated from eq. (34) (red). The abscissa is the normalized longitudinal location, $\bar{z} = z/r_0$ and the ordinate is the ion bunch line number density as calculated from eq. (45). The red and blue curve shows results for zero diffusion. The magenta curve shows the numerical solution of eq. (40)-(42) with finite diffusion coefficient, $D_0 = 100$, and the profile $\bar{D}(\bar{r}) = 1/(1 + \bar{r}^2)$. All curves are plotted for $\bar{t} = 2$.

V. SUMMARY

As shown in Fig. 1, the numerical solution of the Fokker-Planck equation (blue curve) is successfully benchmarked with the analytical results derived in section III (red curve) for zero diffusion coefficients. In the absence of diffusion, a ‘blip’ appears in the central part of the ion bunch as a result of local cooling. However, the ‘blip’ disappears for some finite diffusion coefficient applied in the calculation. The observation that diffusive kick tends to diminish the ‘blip’ is also confirmed by the simulations with macro-particle tracking, as detailed in [4].

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