

MAGNETIC FIELDS IN BULK, FILM, AND MULTILAYER SUPERCONDUCTORS IN FRONT OF A MULTI-TURN COIL *

Takayuki Kubo[†],

KEK, High Energy Accelerator Research Organization, Tsukuba, Ibaraki, Japan
SOKENDAI (the Graduate University for Advanced Studies), Hayama, Kanagawa, Japan

Abstract

The magnetic field distribution formulae in a bulk superconductor, a superconducting film, and an SIS multilayer structure in front of a multi-turn coil are derived, which may be useful for a detailed analysis in a vortex field measurement by using the third harmonic method.

INTRODUCTION

The method of third harmonic analysis is used in some studies to measure vortex penetration fields of superconducting (SC) samples [1, 2]. In this method, a coil much smaller than an SC sample is put in front of the sample as shown in the figure 1. The coil creates an AC magnetic field and induces the AC Meissner screening current on the sample. Then the AC magnetic field generated by the screening current is picked up by the coil. When the sample is in the Meissner state, the picked up voltage shows the sinusoidal curve. When the sample is in the vortex state, the third harmonics appear in addition to the sinusoidal contribution. In the present contribution, in order for our future experimental works using the method of third harmonic analysis, we derive the magnetic field distribution formulae in a bulk SC, an SC film, and an SIS multilayer structure in front of a multi-turn coil.

MODEL AND FORMULATION

We study the model shown in Fig. 2 and investigate its AC response. The coil has an axial symmetric geometry and consists of a discrete distribution of concentric turns [3].

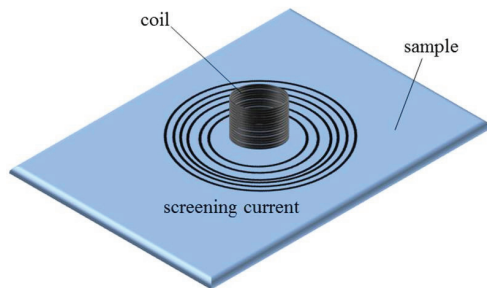


Figure 1: A coil put in front of an SC sample.

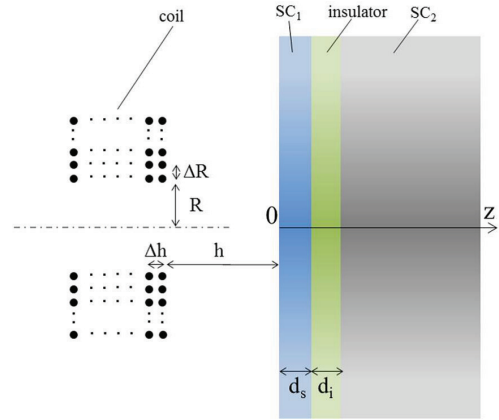


Figure 2: The model examined in the present work.

The radial and axial spacing between adjacent turns are given by ΔR and Δh . The sample surface is located at a distance h from the coil and is parallel to the xy -plane. The thickness of the top SC layer and insulator layer are given by d_s and d_i , respectively, and the SC substrate has an infinite thickness. It should be noted that the sample structure is reduced to a bulk SC when the material SC_1 equals to SC_2 and $d_i \rightarrow 0$ and is reduced to an SC film when $d_i \rightarrow \infty$.

We formulate this model in the cylindrical coordinate. In a following, $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\mathbf{z}}$ represent the unit vector with radial, rotational, and vertical direction, respectively. Considering the axial symmetry of the present model, the vector potential can be written as $\mathbf{A} = A(r, z)\hat{\theta}$. Then the magnetic and electric fields are written as $\mathbf{B} = \text{rot } \mathbf{A} = B_r(r, z)\hat{\mathbf{r}} + B_z(r, z)\hat{\mathbf{z}} = -\partial_z A \hat{\mathbf{r}} + r^{-1} \partial_r(rA) \hat{\mathbf{z}}$ and $\mathbf{E} = i\omega \mathbf{A} = i\omega A(r, z)\hat{\theta}$, respectively, where ω is the angular frequency. In order to evaluate $A(r, z)$, the Maxwell equation,

$$\text{rot rot } \mathbf{A} = \mu_0 \mathbf{j}, \quad (1)$$

must be solved, where \mathbf{j} is a total current density at each point. The current density circulating the coil is given by $j_{\text{coil}}(r, z)\hat{\theta}$ with

$$j_{\text{coil}}(r, z) = I \sum_{n,m} \delta(z - z_n) \delta(r - r_m), \quad (2)$$

where $z_n \equiv -h - n\Delta h$ ($n = 0, 1, 2, \dots$) and $r_m \equiv R + m\Delta R$ ($m = 0, 1, 2, \dots$). The induced screening current density in the SC_p ($p = 1, 2$) region is given by $j_{SC_p} \hat{\theta} = \sigma_p \mathbf{E} = i\omega \sigma_p A(r, z)\hat{\theta}$, where σ_p is the complex conductivity.

* The work is supported by JSPS Grant-in-Aid for Young Scientists (B), Number 26800157, JSPS Grant-in-Aid for Challenging Exploratory Research, Number 26600142, and Photon and Quantum Basic Research Coordinated Development Program from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

[†] kubitaka@post.kek.jp

ity in the SC_p. Then

$$j_{SC_1}(r, z) = -\frac{1}{\mu_0 \ell_1^2} A(r, z) \quad (0 \leq z \leq d_s), \quad (3)$$

$$j_{SC_2}(r, z) = -\frac{1}{\mu_0 \ell_2^2} A(r, z) \quad (z \geq d_s + d_i), \quad (4)$$

where $\ell_p \equiv \sqrt{i/\mu_0 \sigma_p \omega} \simeq \lambda_p$ ($p = 1, 2$) and λ_p is the penetration depth in the SC_p. Then Eq. (1) can be written explicitly in the cylindrical coordinate as follows.

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) A(r, z) = -\mu_0 I \sum_{n,m} \delta(z - z_n) \delta(r - r_m), \quad (5)$$

at $z < 0$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\ell_1^2} \right) A(r, z) = 0, \quad (6)$$

at $0 \leq z \leq d_s$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) A(r, z) = 0, \quad (7)$$

at $d_s < z < d_s + d_i$, and

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{\ell_2^2} \right) A(r, z) = 0, \quad (8)$$

at $d_s + d_i \leq z$. Note that the above differential equations are reduced to those for a single film and a bulk superconductor when $d_i \rightarrow \infty$ and $d_s \rightarrow \infty$, respectively. The boundary conditions are given by the continuity conditions of B_r and B_z at $z = 0, d_s$, and $d_s + d_i$. Finding the solution of Eqs. (5)-(8) is the goal of the next section.

MAGNETIC FIELD DISTRIBUTION

The Hankel transform and the general solution

For solving Eqs. (5)-(8), the Hankel transform is useful. The Hankel transform of a function $f(r)$ and its inverse transform are given by

$$\tilde{f}_\nu(k) = \int_0^\infty r J_\nu(kr) f(r) dr, \quad (9)$$

$$f(r) = \int_0^\infty k J_\nu(kr) \tilde{f}_\nu(k) dk, \quad (10)$$

respectively. Here we introduce an useful relation: the Hankel transform of $[\partial^2/\partial r^2 + (1/r)\partial/\partial r - \nu^2/r^2]f(r)$ is given by

$$\int_0^\infty dr r J_\nu(kr) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{\nu^2}{r^2} \right) f(r) = -k^2 \tilde{f}_\nu(k). \quad (11)$$

Then the Hankel transforms of the both sides of Eqs. (5)-(8) are given by

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{A}(k, z) = -\mu_0 I \sum_{n,m} r_m J_1(kr_m) \delta(z - z_n), \quad (12)$$

at $z \leq 0$,

$$\left(\frac{\partial^2}{\partial z^2} - \beta_1^2 \right) \tilde{A}(k, z) = 0, \quad (13)$$

at $0 \leq z \leq d_s$,

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{A}(k, z) = 0, \quad (14)$$

at $d_s \leq z \leq d_s + d_i$, and

$$\left(\frac{\partial^2}{\partial z^2} - \beta_2^2 \right) \tilde{A}(k, z) = 0, \quad (15)$$

at $d_s + d_i \leq z$, respectively, where Eq. (11) with $\nu = 1$ is used, $\tilde{A}(k, z) \equiv \int_0^\infty dr r J_1(kr) A(r, z)$ is the Hankel transform of $A(r, z)$, and $\beta_p \equiv \ell_p^{-1} \sqrt{1 + k^2 \ell_p^2}$ ($p = 1, 2$). Solving Eqs. (12)-(15), the solution of $\tilde{A}(k, z)$ is given by

$$\tilde{g}(k, z) + C_1(k) e^{kz}, \quad (z \leq 0) \quad (16)$$

$$C_2(k) e^{-\beta_1 z} + C_3(k) e^{\beta_1 z}, \quad (0 \leq z \leq d_s) \quad (17)$$

$$C_4(k) e^{-k(z-d_s)} + C_5(k) e^{k(z-d_s)}, \quad (d_s \leq z \leq d_s + d_i) \quad (18)$$

$$C_6(k) e^{-\beta_2(z-d_s-d_i)}, \quad (d_s + d_i \leq z) \quad (19)$$

where $\tilde{g}(k, z)$ is the Green function that satisfies

$$\left(\frac{\partial^2}{\partial z^2} - k^2 \right) \tilde{g}(k, z) = -\mu_0 I \sum_{n,m} r_m J_1(kr_m) \delta(z - z_n), \quad (20)$$

and C_i ($i = 1, \dots, 6$) are some functions of k .

Derivation of $\tilde{g}(k, z)$

The Fourier transform of the both sides of Eq. (20) yields

$$(-q^2 - k^2) \tilde{g}_F(k, q) = -\mu_0 I \sum_{n,m} r_m J_1(kr_m) e^{iqz_n} \quad (21)$$

or

$$\tilde{g}_F(k, q) = \mu_0 I \sum_{n,m} \frac{r_m J_1(kr_m)}{q^2 + k^2} e^{iqz_n}, \quad (22)$$

where $\tilde{g}_F(k, q) = \int_{-\infty}^\infty \tilde{g}(k, z) e^{iqz} dz$ is the Fourier transform of $\tilde{g}(k, z)$. The inverse Fourier transform of Eq. (22) yields

$$\tilde{g}(k, z) = \frac{\mu_0 I}{2} \sum_{n,m} r_m J_1(kr_m) \frac{1}{k} e^{-k|z-z_n|}, \quad (23)$$

We can easily confirm Eq. (23) satisfies Eq. (20) by using $\partial_z^2 e^{-k|z-z_n|} = -2k\delta(z - z_n) + k^2 e^{-k|z-z_n|}$.

Derivation of $C_i(k)$

The coefficients $C_1(k)$ - $C_6(k)$ can be obtained by imposing the boundary conditions on the general solution. The radial component of the magnetic field is given by

$$B_r(r, z) = -\partial_z A(r, z) = \int_0^\infty dk k J_1(kr) \left(-\frac{\partial \tilde{A}(k, z)}{\partial z} \right). \quad (24)$$

Then the continuity conditions for H_r at $z = 0$, d_s , and $d_s + d_i$ yield

$$\frac{\mu_0 I}{2} \sum_{n,m} r_m J_1(kr_m) \frac{1}{k} e^{kz_n} - C_1 = \frac{\beta_1}{k} (C_2 - C_3) \quad (25)$$

$$C_2 e^{-\beta_1 d_s} - C_3 e^{\beta_1 d_s} = \frac{k}{\beta_1} (C_4 - C_5) \quad (26)$$

$$C_4 e^{-kd_i} - C_5 e^{kd_i} = \frac{\beta_2}{k} C_6. \quad (27)$$

Eqs. (25)-(27) are the first three conditions to determine $C_1(k)$ - $C_6(k)$. The rest three conditions come from the continuity conditions of B_z at $z = 0$, d_s , and $d_s + d_i$. The z -component, B_z , is given by $B_z = r^{-1}(A + r\partial_r A)$ or

$$B_z(r, z) = \int_0^\infty dk k \left(\frac{J_1(kr)}{r} + k J_1'(kr) \right) \tilde{A}(k, z). \quad (28)$$

Then the continuity conditions are given by

$$\frac{\mu_0 I}{2} \sum_{n,m} r_m J_1(kr_m) \frac{1}{k} e^{kz_n} + C_1 = C_2 + C_3 \quad (29)$$

$$C_2 e^{-\beta_1 d_s} + C_3 e^{\beta_1 d_s} = C_4 + C_5 \quad (30)$$

$$C_4 e^{-kd_i} + C_5 e^{kd_i} = C_6. \quad (31)$$

Solving Eqs. (25)-(27) and (29)-(31), we obtain

$$C_1 = \frac{D_+ - \frac{\beta_1}{k} D_-}{D_+ + \frac{\beta_1}{k} D_-} \tilde{g}(k, 0), \quad (32)$$

$$C_2 = \frac{\left(1 + \frac{\beta_2}{\beta_1}\right) \cosh kd_i + \left(\frac{\beta_2}{k} + \frac{k}{\beta_1}\right) \sinh kd_i}{D_+ + \frac{\beta_1}{k} D_-} e^{\beta_1 d_s} \tilde{g}(k, 0), \quad (33)$$

$$C_3 = \frac{\left(1 - \frac{\beta_2}{\beta_1}\right) \cosh kd_i + \left(\frac{\beta_2}{k} - \frac{k}{\beta_1}\right) \sinh kd_i}{D_+ + \frac{\beta_1}{k} D_-} e^{-\beta_1 d_s} \tilde{g}(k, 0), \quad (34)$$

$$C_4 = \frac{1 + \frac{\beta_2}{k}}{D_+ + \frac{\beta_1}{k} D_-} e^{kd_i} \tilde{g}(k, 0), \quad (35)$$

$$C_5 = \frac{1 - \frac{\beta_2}{k}}{D_+ + \frac{\beta_1}{k} D_-} e^{-kd_i} \tilde{g}(k, 0), \quad (36)$$

$$C_6 = \frac{2}{D_+ + \frac{\beta_1}{k} D_-} \tilde{g}(k, 0), \quad (37)$$

where

$$D_+ = \left(\cosh kd_i + \frac{\beta_2}{k} \sinh kd_i \right) \cosh \beta_1 d_s + \frac{k}{\beta_1} \left(\sinh kd_i + \frac{\beta_2}{k} \cosh kd_i \right) \sinh \beta_1 d_s, \quad (38)$$

$$D_- = \left(\cosh kd_i + \frac{\beta_2}{k} \sinh kd_i \right) \sinh \beta_1 d_s + \frac{k}{\beta_1} \left(\sinh kd_i + \frac{\beta_2}{k} \cosh kd_i \right) \cosh \beta_1 d_s. \quad (39)$$

Field distribution in an SC film

The magnetic field distributions in an SC film can be obtained by substituting $d_i = \infty$. Then $D_+ + (\beta_1/k)D_- = (1/2)(1 + \beta_2/k)[2 \cosh \beta_1 d_s + (k/\beta_1 + \beta_1/k) \sinh \beta_1 d_s] e^{kd_i}$, and we obtain

$$C_1^{\text{film}} = \frac{[(k/\beta_1) - (\beta_1/k)] \sinh \beta_1 d_s \tilde{g}(k, 0)}{2 \cosh \beta_1 d_s + [(k/\beta_1) + (\beta_1/k)] \sinh \beta_1 d_s}, \quad (40)$$

$$C_2^{\text{film}} = \frac{[1 + (k/\beta_1)] e^{\beta_1 d_s} \tilde{g}(k, 0)}{2 \cosh \beta_1 d_s + [(k/\beta_1) + (\beta_1/k)] \sinh \beta_1 d_s}, \quad (41)$$

$$C_3^{\text{film}} = \frac{[1 - (k/\beta_1)] e^{-\beta_1 d_s} \tilde{g}(k, 0)}{2 \cosh \beta_1 d_s + [(k/\beta_1) + (\beta_1/k)] \sinh \beta_1 d_s}, \quad (42)$$

$$C_4^{\text{film}} = \frac{2 \tilde{g}(k, 0)}{2 \cosh \beta_1 d_s + [(k/\beta_1) + (\beta_1/k)] \sinh \beta_1 d_s}, \quad (43)$$

$$C_5^{\text{film}} = C_6^{\text{film}} = 0. \quad (44)$$

Field distribution in a bulk SC

The magnetic field distributions in a bulk SC can be obtained by substituting $d_s \rightarrow \infty$ into Eqs (40)-(44).

$$C_1^{\text{bulk}} = \frac{1 - (\beta_1/k)}{1 + (\beta_1/k)} \tilde{g}(k, 0), \quad (45)$$

$$C_2^{\text{bulk}} = \frac{2}{1 + (\beta_1/k)} \tilde{g}(k, 0), \quad (46)$$

$$C_3^{\text{bulk}} = C_4^{\text{bulk}} = C_5^{\text{bulk}} = C_6^{\text{bulk}} = 0. \quad (47)$$

Detailed discussions and applications to experiments will be presented elsewhere.

SUMMARY

The magnetic field distribution in Fig. 2 is given by

$$B_r(r, z) = -\frac{\partial}{\partial z} A(r, z), \quad B_z = \frac{1}{r} \left(1 + r \frac{\partial}{\partial r} \right) A(r, z), \quad (48)$$

with

$$A(r, z) = \int_0^\infty k J_1(kr) \tilde{A}(k, z) dk, \quad (49)$$

where $\tilde{A}(k, z)$ is given by Eqs. (16)-(19), $\tilde{g}(k, z)$ by Eq. (23), and C_i ($i = 1, \dots, 6$) by Eqs. (32)-(37). An SC film and a bulk SC are special cases of Fig. 2; C_i ($i = 1, \dots, 6$) for an SC film and a bulk SC are given by Eqs. (40)-(44) and Eqs. (45)-(47), respectively. Detailed discussions and applications to experiments will be presented elsewhere.

REFERENCES

- [1] G. Lamura, M. Aurino, A. Andreone, and J. -C. Villegier, J. Appl. Phys. **106**, 053903 (2009).
- [2] C. Z. Antoine, J.-C. Villegier, and G. Martinet, Appl. Phys. Lett. **102**, 102603 (2013).
- [3] M. Aurino, E. Di Gennaro, and F. Di Iorio, J. Appl. Phys. **98**, 123901 (2005).