INTEGRATED GREEN FUNCTION FOR CHARGED PARTICLE MOVING ALONG BENDING ORBIT

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Abstract

Electro-magnetic field for moving charged particle is given by Liennard- Wiechert potential. Three dimensional electro-magnetic field near a particle with a given trajectory is calculated on grid space. The field is regarded as wake field, which makes possible to evaluate behavior of entire bunch. Effcts of coherent synchrotron radiation and/or space charge force for relativistic beam can be studied by Green function.

INTRODUCTION

A moving charged particle with position and velocity, $\boldsymbol{x},\boldsymbol{\beta}=d\boldsymbol{x}/d(ct)$ experiences Lorentz force from another (source) charged particle at $\boldsymbol{x}',\boldsymbol{\beta}'=d\boldsymbol{x}'/d(ct')$,

$$\boldsymbol{F}=e(\boldsymbol{E}+c\boldsymbol{\beta}\times\boldsymbol{B})=e\left[\boldsymbol{E}+\boldsymbol{\beta}\times(\boldsymbol{n}\times\boldsymbol{E})\right] \quad (1)$$

The electro-magnetic field induced by the source particle is given by [1]

$$\boldsymbol{E} = \frac{e}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{n}-\boldsymbol{\beta}'}{\gamma^2\kappa^3R^2}+\frac{\boldsymbol{n}\times((\boldsymbol{n}-\boldsymbol{\beta}')\times\boldsymbol{\alpha}')}{\kappa^3R}\right] \quad (2)$$

$$\boldsymbol{B} = \frac{1}{c}\boldsymbol{n}\times\boldsymbol{E}. \quad (3)$$

where $\boldsymbol{R}=\boldsymbol{x}-\boldsymbol{x}'$, $R=|\boldsymbol{R}|$, $\boldsymbol{n}=\boldsymbol{R}/R$ and $\kappa=1-\boldsymbol{n}\cdot\boldsymbol{\beta}'$. The relation of the times is expressed by

$$t=t'+\frac{R}{c}. \quad (4)$$

In accelerator society, time variable is the position along beam line, s. Longitudinal variable is difference of arrival time for a reference particle, $z=\beta c(t_0-t)=s-\beta ct$, where $t=0$ is arrival time of the reference particle, $s=\beta ct_0$. Canonical momentum for z is $\Delta p/p_0$. The canonical momenta (p_x,p_y) are normalized by $p_0=m\beta_0\gamma_0 c$. Electro-magnetic field at s is induced by the source particle at a different location, s', which is determined by the time relation of Eq.(4). The time relation Eq.(4) is translated to

$$s=s'+\beta R(x,y,s-s')+z. \quad (5)$$

Lienard-Wiechert potential, which specifies electro-magnetic field of single particle, has a role of Green function. For given charge distribution, Lorentz force, which a particle with $\vec{x}=(x,y,z)$ and $\vec{p}=(p_x,p_y,\delta)$ experiences, is given by

$$\bar{\boldsymbol{F}}(\vec{x},\vec{p},s)=\int \boldsymbol{F}(\vec{x},\vec{p},s;\vec{x}',\vec{p}'s')\Psi(\vec{x}',\vec{p}';s')d\vec{x}'d\vec{p}' \quad (6)$$

where Ψ is distribution of beam particles in phase space. s' is satisfied to Eq.(5) and $p_{x,y}=\beta_{x,y}\gamma/(\beta_0\gamma_0)\approx\beta_{x,y}$. We assume particles move the same trajectory with parallel displacement in a traveling distance Δs. Energy distribution of the source particle is not considered. The distribution is characterized by only $\vec{x}$ with the projection of the real space, $\psi(\vec{x})=\int\Psi d\vec{p}$, and is kept for the traveling Δs. This assumption is valid for $\Delta s<\beta_{xy}$, where β_{xy} is the beta function in an accelerator optics. Integrating Δs, momentum/energy change is calculated by

$$\Delta\vec{p}=\frac{Nr_e}{\gamma}\int\vec{W}(\vec{x};\vec{x}')\psi(\vec{x}')d\vec{x}' \quad (7)$$

where integral for s is done for only $\boldsymbol{F}$,

$$\vec{W}(\vec{x};\vec{x}')=\left(\frac{e^2}{4\pi\varepsilon_0}\right)^{-1}\int_{\Delta s}\boldsymbol{F}(\vec{x},s;\vec{x}',s'))ds \quad (8)$$

$\boldsymbol{F}$ is independent of s, when particle move sufficiently inside from entrance of a bending magnet, so-called in "stationary case". $\boldsymbol{F}$'s near entrance and exit are evaluated and integrated step by step along s. Since particles move along parallel displaced trajectory, Lorentz force has translation symmetry, $\boldsymbol{F}(\vec{x},s;\vec{x}',s')=\boldsymbol{F}(\vec{x}-\vec{x}',s)$, therefore $W(\vec{x};\vec{x}')=W(\vec{x}-\vec{x}')$.

Particle In Cell (PIC) simulation is popularly used to study variation of the beam distribution. Macro-particle distribution is mapped on a meshed space. Force, which particles experience, are evaluated by Eq.(8), where the integral is performed on meshed space and Green function is better to be integrated in each mesh area, $(x_i\pm\Delta x/2, y_j\pm\Delta y/2, z_k\pm\Delta z/2)$.

$$\boldsymbol{F}(\vec{x}_{ijk},s;0,0)=\int\int\int_{\Delta x\times\Delta y\times\Delta z}\boldsymbol{F}(\vec{x},s;0,0)d\vec{x}. \quad (9)$$

The integrated Green function (IGF) gives Lorentz force induced by uniform charge distribution in the area, or in different word, averaged Lorentz force induced by single charge. Higher frequency component than $1/\Delta x,y,z$ are cut off, while numerical noise of high frequency component is reluxed. The area size depends on which frequency range is essential in target phenomenon.

IGF was used to calculate beam-beam force interacting in colliders. For lepton colliders, beam aspect ratio is very large $\sigma_x/\sigma_y\approx 100$ at collision point; flat beam collision. IGF was essential to obtain an accurate beam-beam force for flat beam [2–4]. Here IGF is used to cut high frequency component.

This paper is devoted to calculate the integrated Green function for Lorentz force moving in a bending magnet in

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near field region. Lorentz force is singular for high γ. The integration has to be done carefully with taking into account the singular behavior.

ELECTRO-MAGNETIC FIELD NEAR SOURCE PARTICLE IN BENDING MAGNET

Electro-magnetic field for the stationary case is discussed here [5–7]. Field at entrance and exit is discussed in similar way and is discussed elsewhere. Figure 1 shows geometrical relation in bending magnet. A source particle moves reference orbit and arrives at s. Electro-magnetic field is calculated near the source particle using Eq.(2). Coordinates (x', y', ζ') for the source particle are obtained on the system defined at s as shown in Fig. 1. Motion of source particle $(\boldsymbol{x}', \boldsymbol{\beta}', \boldsymbol{\alpha}')$ is represented as function of $\theta = (s - s')/\rho$.

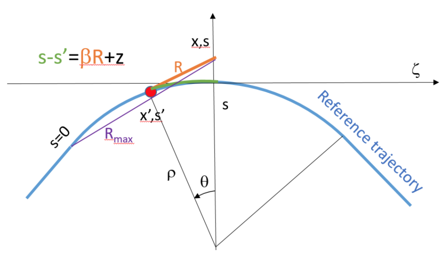

Figure 1: $x'(z), \zeta(z')$.

Motion of the source particle is expressed by

$$x' = \rho(\cos\theta - 1) \qquad \zeta' = -\rho\sin\theta \tag{10}$$

$R_\rho = R/\rho$ is given by

$$R_\rho(x, y, \theta) = \sqrt{\zeta'^2 + (x - x')^2 + y^2}/\rho = \sqrt{4(1 + x_\rho)\sin^2\frac{\theta}{2} + x_\rho^2 + y_\rho^2}. \tag{11}$$

where $(x, y, z)_\rho = (x, y, z)/\rho$. z_ρ is represented as function of (x_ρ, y_ρ, θ) by,

$$z_\rho = \theta - \beta\sqrt{4(1 + x_\rho)\sin^2\frac{\theta}{2} + x_\rho^2 + y_\rho^2}. \tag{12}$$

$\kappa = dz_\rho/d\theta$ is expressed by

$$\kappa = 1 - \boldsymbol{n}\cdot\boldsymbol{\beta} = 1 - \frac{\beta(1 + x_\rho)\sin\theta}{R_\rho}. \tag{13}$$

Lorentz force, Eqs.(1) to (3), are represented as functions of (x_ρ, y_ρ, θ). Radiation part, 2nd term of Eq.(2), is expressed by

$$E_s^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3 \kappa^3}\left[2\sin^2\frac{\theta}{2}(R_\rho\beta - \sin\theta) + \left\{R_\rho\beta - \sin\theta(2 - \cos\theta)\right\}x_\rho - (x_\rho^2 + y_\rho^2)\sin\theta\right]. \tag{14}$$

$F_s = E_s$, because of $B_s = 0$. Lorentz force for horizontal and vertical is expressed by

$$F_x^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3\kappa^3}\left[\sin^2\theta + 4\beta^2\sin^2\frac{\theta}{2} - 2R_\rho\beta\sin\theta + \left(\sin^2\theta + 4\beta^2\sin^2\frac{\theta}{2} - R_\rho\beta\sin\theta\right)x_\rho + \beta^2 x_\rho^2 + y_\rho^2\cos\theta\right] \tag{15}$$

$$F_y^{(r)} = \frac{\beta^2}{\rho^2 R_\rho^3\kappa^3}\left[2\sin^2\frac{\theta}{2}(1 + \beta^2) - R_\rho\beta\sin\theta + \left(2\sin^2\frac{\theta}{2} - \frac{1}{\gamma^2}\right)\right]y_\rho. \tag{16}$$

The force is calculated for SACLA beam extraction, E = 8 GeV, ρ = 46.4 m, B = 0.57 T. Figure 2 shows $\boldsymbol{F}^{(r)}$, as function of z. $F_s^{(r)}(0,0,0)$ in Fig. 2(a) agrees with the formula $-2\gamma^4/\rho^2 = -5.6\times10^{13}$ m^{-2} (deceleration). $F_s^{(r)}(x, 0, z > 0.01\text{nm})$ is positive (acceleration) and is weakly dependent on x. $F_s^{(r)}$ has different behavior for positive or negative x also shown in Fig. 7 of Ref [6]. $F_x^{(r)}$ is positive (repulsive) and is not symmetric for the sign of x [8]. $F_y^{(r)}$ is negative(attractive).

Coulomb term, which is 1st term in Eq.(2), is calculated by similar way. They are very weak compare with the radiation terms at E=8 GeV.

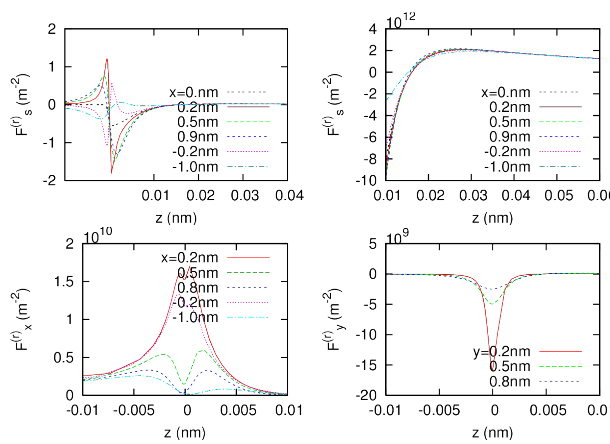

Figure 2: Lorentz force near a point charge. (a) $F_s^{(r)}$, (b) $F_y^{(r)}$ and (c) $F_y^{(r)}$.

INTEGRATED GREEN FUNCTION

Integrated Green function is obtained by integration in area $\Delta x\times\Delta y\times\Delta z$ nearby (x_i, y_j, z_k),

$$\bar{\boldsymbol{F}}(x_i, y_j, z_k) = \int\int\int_{\Delta x\times\Delta y\times\Delta z}\boldsymbol{F}(x, y, \theta)dzdxdy \tag{17}$$

Lorentz force for a single charge is singular at $\vec{x} = 0$, but its for a charge distribution with a finite density is regular. For beam-beam or space charge force, we have analytic form for the integration [3]. The formulae for Lienard-Wiechert potential Eqs.(14)-(16) are too complex, thus numerical integration is performed. The numerical integral should be converge, since the integral is regular.

Lorentz force/Green function inside a mesh area is integrated with taking into account the micro-scopic behavior shown in Fig. 2. Green function contains high frequency component of $\lambda_c = 4\pi\rho/(3\gamma^3)$ near $\vec{x} = 0$. Integration step is determined so as to take into account the high frequency component. The steps are chosen to be $ds = a\gamma^2\lambda_c$, $dz = a\lambda_c$, $dx, dy = a\gamma\lambda_c$, where a is a factor for the step size. Figure 3 shows convergence of the integrated Green Function of nearest mesh at $\vec{x} = 0$, $x = 0 \sim \Delta x$, $y = 0 \sim \Delta y$, $z = 0 \sim \Delta z$. Convergence for meshes far from $\vec{x} = 0$ is better. The integrated Green function is given with a sufficient accuracy ($< 1\%$) for $a < 0.1$.

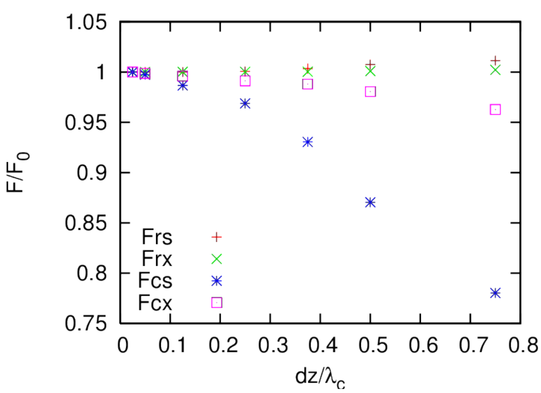

Figure 3: Convergence of Integrated Green function at for integration step.

We calculated integrated Green function using $a = 0.01$ with a safety margin. Figure 4 shows the integrated Green function for $F_s^{(r)}$ along z. Three points are given for $x = 2.5, 7.5, -7.5$ μm. Plots (b) focus to positive z area; acceleration force for leading particles. The plot (b) agrees with analytical formula (dashed line) perfectly. Lorentz force is dependent on x for $z = \pm 0.05$ μm, but is independent for $z > 0.1$ μm. The same behavior is seen also in micro-scopic as shown in Fig. 2. Figures 5 and 6 shows integrated Green function for the transverse radiation part, $F_x^{(r)}$ and $F_y^{(r)}$, respectively. Plots (a) and (b) is downstream ($z < 0$) and upstream ($z > 0$) of the source particle. The horizontal force for negative z in plot (a) has singular behaviors at positive x. The behaviors are seen in the tangential direction of the orbit. The vertical force is 1 or 2 orders smaller than horizontal ones. It is focusing for negative z, but defocusing for positive z.

Figure 7 shows Lorentz force convoluted by beam distribution, $N_e = 1.7 \times 10^9$ (270 pC). The angular divergence of the beam is $\sigma_{p_x} = 2.3\mu$rad. The transverse force, which is sexupolar, is not negligible compare with the beam angular divergence.

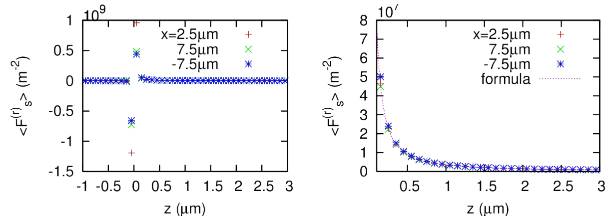

Figure 4: Integrated Green function at for integration step.

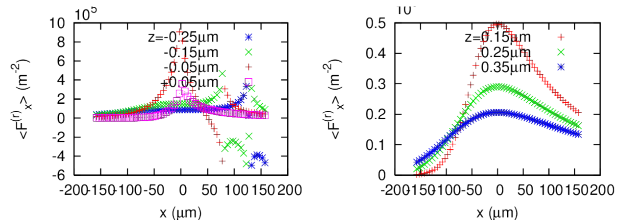

Figure 5: Integrated Green function at for integration step.

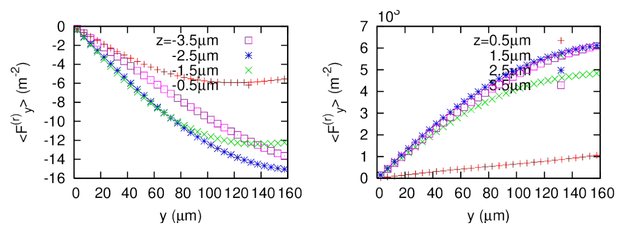

Figure 6: Integrated Green function at for integration step.

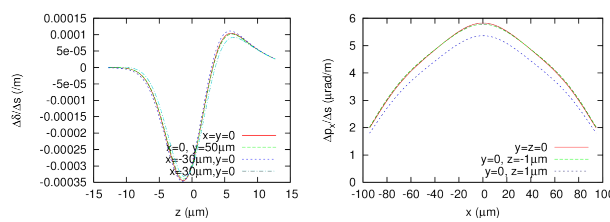

Figure 7: Convoluted beam kick. Sum. of radiation and Coulomb force. SACLA final branch line.

CONCLUSIONS

Integrated Green function in bending magnet was obtained from single charge Liennard-Wiechert field. 3 dimentional wake force due to synchrotron radiation was estimated. The longitudinal force has been studied for a long time. The transverse force did not seem negligible for SACLA final branch to multi-beam line.

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