NOVEL IMPLEMENTATION OF QUADRUPOLE AND HIGHER ORDER FRINGE FIELDS TO ACCELERATOR DESIGN

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Abstract

Until recently, in the initial design phase of any accelerator project, it was not possible to have an adequate description of quadrupole and higher order multipole fringe fields. We report on the latest developments in analytical fringe fields for multipoles, particularly for quadrupoles and sextupoles. We show how they can be used to improve accelerator codes and make them both faster and more precise. We also show how the analytical formulae for the fringe fields yield expressions for both the scalar and vector potentials in electromagnetism. We conclude by discussing the application of both potentials to the design of multipole magnets as well as the implementation of symplectic kick approximations for fringe fields in thin lens models that could be used in accelerator codes.

INTRODUCTION

It is very important to consider fringe fields of low order multipoles in the early stages of the design of an accelerator. In particular, it is useful to implement these into accelerator codes as well. This means that, for example in time-domain space charge tracking codes such as the General Particle Tracer (GPT) code [1], computing time is not wasted on discontinuities due to the hard edge models people predominantly use to describe multipoles and everything is smooth instead. This not only makes the codes considerably faster, as illustrated in [2], but also helps with the accuracy of the tracking by including an approximation of effects which exist in real accelerators because all magnets have fringe fields.

We present below a simplified version of the expressions for fringe fields for multipoles found in [3] for the case of quadrupoles and sextupoles, together with a brief summary of the derivation. The expressions for the field in the quadrupole case were already presented in [2, 3] and are repeated here together with a brief explanation of their origin as well as the scalar and vector potentials associated with them. The expressions for the fringe fields presented are a simplification because, unlike the more general results presented in [3], it is no longer possible to control the behaviour of the fringe field as one goes off-axis transversally, with respect to the longitudinal coordinate, at the exit of the quadrupole or sextupole. However, this should not be an issue unless a particular type of accelerator is being designed, for example, EMMA [4, 5], where one of the main aspects of a non-scaling FFAG is that the particle trajectories go considerably far away from the axis due to the nature of these accelerators. But, even in this case, the effect should be almost negligible.

FRINGE FIELDS FOR DIPOLES

To illustrate the solutions of Maxwell's equations, we look at dipoles first as the general case for any order multipole can be considered a relatively simple generalisation of this. For dipoles, it is sufficient to consider a two dimensional version of the magneto-static equations $\vec{\nabla} \times \vec{B} = \vec{\nabla} \cdot \vec{B} = 0$, taking $B_x = 0$, these simplify to:

$$\partial_y B_y + \partial_z B_z = \partial_y B_z - \partial_z B_y = 0, \tag{1}$$

and

$$\partial_x B_z = \partial_x B_y = 0,$$

so there is no dependence on x. In this paper, we also consider fringe fields, for both dipoles and higher order multipoles, with an Enge-type fall-off [6], though others are possible [7]

$$F(z) = \frac{1}{1 + \exp\left[E(z)\right]},$$

with $E(z)$ given by

$$E(z) = a_1 + a_2\left(\frac{z}{D}\right) + a_3\left(\frac{z}{D}\right)^2 + \ldots + a_6\left(\frac{z}{D}\right)^5,$$

with all a_i constants determined by models and/or experiment. An advantage of the Enge-type decay is that the fields can be made to decay almost arbitrarily rapidly. The same cannot be said, for example, of an arctan type fall-off [8]. Maxwell's Equations (1), through cross differentiation, imply:

$$\Delta_{y,z} B_y = \Delta_{y,z} B_z = 0,$$

where $\Delta_{y,z} = \partial_y^2 + \partial_z^2$. Both equations can be easily solved (for B_y and B_z) to give $B_y = e(z + iy) + f(z - iy)$ and $B_z = g(z + iy) + h(z - iy)$. If we further ask that Equations (1) be solved as well and we restrict ourselves to real fields, we obtain:

$$B_y = e(z + iy) + \bar{e}(z - iy), \tag{2}$$

$$B_z = -ie(z + iy) + i\bar{e}(z - iy). \tag{3}$$

Applying the equations derived above for dipoles with Enge-type fall-offs, we have

$$B_y = \frac{1}{2(1 + e^{E(z+iy)})} + \frac{1}{2(1 + e^{E(z-iy)})},$$

which would force B_z to have the form

$$B_z = \frac{-i}{2(1 + e^{E(z+iy)})} + \frac{i}{2(1 + e^{E(z-iy)})},$$

for some complex function $E(z + iy)$. If we consider the simple case $E(z + iy) = z + iy$, so with just the a_2 Enge coefficent non-zero and normalised to one, then expressions simplify to:

$$B_y = \frac{(1+e^z\cos(y))}{1+2e^z\cos(y)+e^{2z}}, B_z = \frac{-e^z\sin(y)}{1+2e^z\cos(y)+e^{2z}}.$$

This may be extended to include as many parameters of the Enge function as desired, the only restriction being that $E = E(z+iy)$. Having said this, the above representation has the additional benefit that, from a computational point of view and provided the fringe field is positioned in such a way that the $z = 0$ point is *precisely* where the edge of the hard edge magnet used to be, the overall integrated length of the magnet is the same as that of the original hard edge. This is due to the chosen decay having a π rotational symmetry about the $z = 0$ point.

EXTENSION TO HIGHER ORDER MULTIPOLES

In order to extend the fringe fields to higher order multipoles and to arrive at expressions which are similar in nature to Equations (2,3), it is convenient to introduce the complex coordinates $u = \frac{1}{\sqrt{2}}(x+iy)$ and $v = \frac{1}{\sqrt{2}}(x-iy)$ and to define the transformation / rescaling of Maxwell's equations: $B_u = \frac{1}{\sqrt{2}}(B_x + iB_y)$, $B_v = \frac{1}{\sqrt{2}}(B_x - iB_y)$ and $B_\zeta = \frac{1}{\sqrt{2}}B_z$, $\zeta = \frac{1}{\sqrt{2}}z$, so we have:

$$\partial_u B_u + \partial_\zeta B_\zeta = \partial_z B_u - \partial_v B_\zeta = 0,$$

together with their complex conjugates. From which one can see immediately that, in the absence of any fringe fields, the general solution of Maxwell's equations for any magnet, acting transversely only and without fringe ($B_z = 0$) is given by $B_u = f(v)$ and $B_v = h(u)$ for some functions f and h. The case of a multipole is given by $B_u = iv^n$, $B_v = -iu^n$ and $B_\zeta = 0$, so a quadrupole is $n = 1$, $B_u = iv$, $B_v = -iu$ and $B_\zeta = 0$ and so on. In this way and through a relatively simple extension of the solution for the dipole case, together with an extensive use of the fact that the Maxwell equations are linear, it is possible to add as many solutions as desired together and to rescale them as well, as detailed in [3].

Therefore, the full solution for a multipole of order n with Enge-type fall-off of the field can be written as:

$$\begin{aligned}
B_u &= \sum_{j=1}^{n+1} ib_jc_j\left[(\zeta+ih_j)^nF(n;\zeta+ih_j)\right.\\
&\quad\left.-(-1)^{n+1}(\zeta-ih_j)^nF(n;\zeta-ih_j)\right],\\
B_v &= \sum_{j=1}^{n+1} i\frac{c_j}{b_j}\left[(\zeta+ih_j)^nF(n;\zeta+ih_j)\right.\\
&\quad\left.-(-1)^{n+1}(\zeta-ih_j)^nF(n;\zeta-ih_j)\right],\\
B_\zeta &= \sum_{j=1}^{n+1} c_j\left[(\zeta+ih_j)^nF(n;\zeta+ih_j)\right.\\
&\quad\left.+(-1)^{n+1}(\zeta-ih_j)^nF(n;\zeta-ih_j)\right],
\end{aligned}$$

where the constants b_j and c_j satisfy various relationships [3], $h_j = \frac{u}{b_j} + b_jv$ and we have chosen for all j:

$$\begin{aligned}
F_j(\zeta+ih_j) &= F(n;\zeta+ih_j),\\
G_j(\zeta-ih_j) &= (-1)^{n+1}F(n;\zeta-ih_j).
\end{aligned}$$

The functions $F(n;\xi)$ with complex argument ξ are constructed in a way that the multipole gradient has the usual Enge-type fringe field fall-off, so

$$F(0;\xi) = \frac{1}{1+e^\xi}.$$

and the functions $F(n;\xi)$ for $n > 0$ are then obtained inductively and through repeated integration [3] so that, for any (positive integer) n and (real argument) ζ, $F(n;\zeta)$ can be written in the form:

$$F(n;\zeta) = 1 + \frac{n!}{\zeta^n}\mathrm{Li}_n(-e^\zeta) - \sum_{j=1}^{n}\frac{n!}{(n-j)!\zeta^j}\mathrm{Li}_j(-1),$$

where $\mathrm{Li}_n(\zeta)$ is the polylogarithm (or Jonquière function [9]) of order n. It is possible to let all constants $b_i \to 1$ for every order of multipole, in what follows we show the results of this in detail for quadrupoles and sextupoles.

FRINGE FIELDS FOR QUADRUPOLES

When we take limits of the coefficients b_1 and b_2 as they both tend to 1 in the above expressions, we obtain the following for the components of the field:

$$\begin{aligned}
B_x &= \frac{1}{8}\left(\frac{2(4e^{i\sqrt{2}x}+2e^{\sqrt{2}(2z+ix)}+3e^{\sqrt{2}z}(1+e^{2i\sqrt{2}x}))y}{(e^{i\sqrt{2}x}+e^{\sqrt{2}z})(1+e^{\sqrt{2}(z+ix)})}\right.\\
&\qquad\left.+i\sqrt{2}\ln\left[\frac{1+e^{\sqrt{2}(z+iy)}}{1+e^{\sqrt{2}(z-iy)}}\right]\right),\\
B_y &= B_x(x\leftrightarrow y),\\
B_z &= \frac{1}{8}\left\{-y\left(\tan\left[\frac{x-iz}{\sqrt{2}}\right]+\tan\left[\frac{x+iz}{\sqrt{2}}\right]\right)\right.\\
&\qquad\left.-x\left(\tan\left[\frac{y-iz}{\sqrt{2}}\right]+\tan\left[\frac{y+iz}{\sqrt{2}}\right]\right)\right\}.
\end{aligned}$$

There are several ways this can be done. The easiest is to use a program such as Mathematica [10] and simply take a limit of these constants to the desired value. Alternatively and because b_1 and b_2 depend on each other so there is only really one variable, it is possible to use L'Hôpital's rule (because the original expressions exhibit singularities) to obtain the same result. Note that several expressions for the fringe field of a quadrupole can be found [2, 3], all are equivalent when the various exponentials are converted to trigonometric functions and vice versa. As discussed in [3] for the case of quadrupoles the fringe fields given above have been constructed to have a symmetry under rotation by $\pi/2$ about the z axis, just like the inside of the magnet. However, in the interest of simplicity, this will not be done in the case of sextupoles, which should have a $\pi/3$ symmetry,

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as this makes the expressions unwieldy. Moreover, it only really has an effect when one goes very far away from the central longitudinal axis, where even the beam in non-scaling FFAGs does not go.

FRINGE FIELDS FOR SEXTUPOLES

For the case of sextupoles, things are different and, instead of just having to take the limit of one constant, there are now two. More precisely, there are three constants, b_1, b_2 and b_3 but one depends on the other two. The traditional version of L'Hôpital's rule no longer applies, however, one can still compute the limits in Mathematica with the result:

$$\begin{aligned} B_x &= \frac{1}{2} y \left(2\sqrt{2}x + i \ln\left[1+e^{\sqrt{2}(z+ix)}\right] \right. \\ &\left. -i \ln\left[1+e^{\sqrt{2}(z-ix)}\right]\right), \\ B_y &= \frac{1}{2\sqrt{2}} \left(2x^2 - \mathrm{Li}_2\left[-e^{\sqrt{2}(z-ix)}\right] - \mathrm{Li}_2\left[-e^{\sqrt{2}(z+ix)}\right] \right. \\ &\left. -2y^2 + \mathrm{Li}_2\left[-e^{\sqrt{2}(z-iy)}\right] + \mathrm{Li}_2\left[-e^{\sqrt{2}(z+iy)}\right]\right), \\ B_z &= \frac{1}{4}\left\{2y\left(\ln\left[1+e^{\sqrt{2}(z+ix)}\right] + \ln\left[1+e^{\sqrt{2}(z-ix)}\right]\right)\right. \\ &\left. +i\sqrt{2}\left(\mathrm{Li}_2\left[-e^{\sqrt{2}(z-iy)}\right] - \mathrm{Li}_2\left[-e^{\sqrt{2}(z+iy)}\right]\right)\right\}, \end{aligned}$$

and verify that all the required equations are satisfied. It should be possible to recreate this via a multivariate version of L'Hôpital's rule [11, 12] as well.

SCALAR AND VECTOR POTENTIALS

There are many applications for which it is useful to consider the scalar and vector potentials, ϕ and $\vec{A}$ with $\vec{B} = \vec{\nabla}\phi = \vec{\nabla} \times \vec{A}$, for the fringe fields of the multipole magnets considered. For example, for an iron dominated magnet, the surfaces of constant scalar potential ϕ correspond to the various possible pole faces of the magnet. This was shown in [3] for the scalar potential of a quadrupole ϕ_q. For the limiting case inwhich the coefficients b_1 and b_2 tend to 1, we give the scalar potential for a quadrupole and a sextupole below. Simlarly, the vector potential is also readily available from the expressions given in [3]. The usual limiting procedure can also be taken in this case and the results are given below for both quadrupoles and sextupoles. The vector potential can be used for symplectic tracking through fields as described in [13].

Quadrupole Case

The scalar potential for a quadrupole, ϕ_q, is given by:

$$\begin{aligned} \phi_q = \frac{1}{8}&\left(8xy + i\sqrt{2}y \ln\left[\frac{1+e^{\sqrt{2}(z+ix)}}{1+e^{\sqrt{2}(z-ix)}}\right]\right. \\ &\left. +i\sqrt{2}x \ln\left[\frac{1+e^{\sqrt{2}(z+iy)}}{1+e^{\sqrt{2}(z-iy)}}\right]\right), \end{aligned}$$

whereas the vector potential, $\vec{A}_q$, is given by:

$$\begin{aligned} A_{qx} &= \frac{1}{8}\left(8xz + \sqrt{2}x\left\{\ln 16\right.\right. \\ &\left. -\ln\left[\left(1+e^{\sqrt{2}(z-iy)}\right)\left(1+e^{\sqrt{2}(z+iy)}\right)\right]\right\} \\ &\left. +i\mathrm{Li}_2\left[-e^{\sqrt{2}(z-ix)}\right] - i\mathrm{Li}_2\left[-e^{\sqrt{2}(z+ix)}\right]\right), \\ A_{qy} &= -A_{qx}(x \leftrightarrow y), \\ A_{qz} &= 0. \end{aligned}$$

Sextupole Case

It is possible to give similar expressions for ϕ_s and $\vec{A}_s$ in the case of a sextupole. We point out once again that the equations given below do not possess the property that the fringe has a complete $\pi/3$ rotational symmetry in the fringe field region. However, they do lead to fields which satisfy Maxwell's equations and are easier to implement in any computer code than their symmetrised counterpart.

$$\begin{aligned} \phi_s &= \frac{1}{12}\left\{-2\sqrt{2}y^3 + 3\sqrt{2}y\left(2x^2 - \mathrm{Li}_2\left[-e^{\sqrt{2}(z-ix)}\right]\right.\right. \\ &\left. -\mathrm{Li}_2\left[-e^{\sqrt{2}(z+ix)}\right]\right) + 3i\left(\mathrm{Li}_3\left[-e^{\sqrt{2}(z-iy)}\right]\right. \\ &\left.\left. -\mathrm{Li}_3\left[-e^{\sqrt{2}(z+iy)}\right]\right)\right\}, \\ A_{sx} &= \frac{1}{4}\left\{\left(x^2-y^2\right)\left(2\sqrt{2}z + \ln 4\right) - \mathrm{Li}_3\left[-e^{\sqrt{2}(z-ix)}\right]\right. \\ &-\mathrm{Li}_3\left[-e^{\sqrt{2}(z+ix)}\right] + \mathrm{Li}_3\left[-e^{\sqrt{2}(z-iy)}\right] \\ &\left. +\mathrm{Li}_3\left[-e^{\sqrt{2}(z+iy)}\right]\right\}, \\ A_{sy} &= -\frac{1}{4}y\left\{4x\left(\sqrt{2}z + \ln 2\right) + i\sqrt{2}\left(\mathrm{Li}_2\left[-e^{\sqrt{2}(z-ix)}\right]\right.\right. \\ &\left.\left. -\mathrm{Li}_2\left[-e^{\sqrt{2}(z+ix)}\right]\right)\right\}, \\ A_{sz} &= 0. \end{aligned}$$

CONCLUSIONS

Analytical expressions for the fringe fields of quadrupoles and sextupoles were given. These have already been applied to the code GPT with significant advantages over previous representations of the fringe region [2] and it is hoped that they will be useful in other codes as well. When applied to time-based computer codes they should facilitate considerably the accuracy and computing speed with which particles are tracked. Possible further work could include determining exactly what kind of fringe field behaviour is desired at the start of a project rather than after the magnets have been delivered as well as the application of the vector potential to the symplectic tracking of particles through quadrupole and sextupole fringe fields.

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