ENTRANCE AND EXIT CSR IMPEDANCE FOR NON-ULTRARELATIVISTIC BEAMS*

R. Li, Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA
C.-Y. Tsai, SLAC, Stanford, CA 94025, USA

Abstract

Analytical expression of CSR impedance is necessary for the study of the CSR-induced microbunching instability. In this paper, we present analysis of the transient CSR impedance of the entrance and exit cases for the non-ultrarelativistic regime, and show that the new results reduce to the existing expressions at the ultrarelativistic limit.

INTRODUCTION

The CSR-induced microbunching instability (μBI) can be studied either in the time domain by particle tracking [1, 2] or in the frequency domain by the Vlasov analysis [3-5]. Particle tracking is usually based on the CSR wakefield calculation. The validity of the Vlasov analysis can be verified by benchmarking the micro-bunching gain results from the semi-analytical Vlasov solver against those extracted from careful particle tracking [6]. Such benchmarking can be done only when the CSR wakefield used in particle tracking and the CSR impedance used in Vlasov analysis are consistent with each other for a given regime of approximations.

Most particle tracking codes for machine designs use analytical expressions of CSR wakefield based on the 1D rigid-bunch model. For example, ELEGANT uses the ultra-relativistic limit $(\gamma \to \infty)$ of the CSR wakefield expressions from Ref. [7], for the steady-state [8] as well as the (entrance and exit) transient CSR interactions [9]. The analyses of CSR wakefields are later extended to a wider energy regime [10,11], with the CSR shielding effect included in Ref. [11]. Likewise, for the Vlasov μBI analysis, the steady-state CSR impedance in the ultrarelativistic regime was employed in early studies [3,4], which is later extended to include entrance-CSR impedance at $\gamma \to \infty$ [12]. Further applications of the μBI theory to lower energy or shorter perturbation wavelength are made possible by extending the steady-state CSR impedance to the non-ultrarelativistic (or finite γ) regime [13]. In this paper, we continue to expand our previous study of the steady-state CSR impedance for finite γ to the transient CSR regime, including both the entrance and exit interactions. At $\gamma \to \infty$, our results reduce to the ultrarelativistic results of the existing theories.

ENTRANCE CASE

The wakefield of CSR interaction for a bunch entering a magnetic dipole can be studied based on the Lienard-Wiechert (LW) field between two-particles. Unlike the previous approach [7] that applies Taylor expansion for the LW fields, here we give the exact analytical expression of the CSR wakefield and impedance by using the relation between LW fields and potentials, based on an early study of the entrance-CSR wakefield [14]. In the following we describe the geometry of the problem and briefly summarize the results.

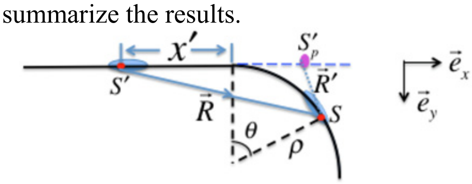

Figure 1: Interaction from S' to S at dipole entrance.

Consider a rigid-line bunch, with design energy $E=\gamma mc^2$, moving in free space from a straight path to a circular orbit of radius ρ (see Fig. 1). Let s be the pathlength parameter along the design orbit, z the longitudinal distance of a particle from the bunch center, and $\lambda(z)$ the density distribution of the bunch. The motion of a particle, with intra-bunch coordinate z, is described by $s=z+\beta ct$ for $\beta=\sqrt{1-\gamma^{-2}}$. Here we set $t=0$ for the moment when the bunch center $(z=0)$ is at the entrance of the dipole $(\theta=0)$.

At time t, the wakefield of the bunch on a test particle S, at angle θ on the circular orbit, is the integration of LW fields generated from all other particles in the bunch

$$E_s(\theta,t)=\int_{-\infty}^{z_c'} E_{s0}^{(A)}(\theta,t;z')\lambda(z')dz'+\int_{z_c'}^{\infty} E_{s0}^{(B)}(\theta,t;z')\lambda(z')dz'. \quad (1)$$

Here $E_{s0}(\theta,t;z')$ is the longitudinal electric field reaching S at $(s=\rho\theta,t)$, which is emitted from the source particle S' at the retarded location and time (s',t') with $z'=s'-\beta ct'$. Adopting notations in Ref. [7], we use the superscript (A) for the case when S' is on the straight path $(s'=-x')$ and (B) for S' on the circular orbit $(s'=\rho\theta')$. For the integration limits in Eq. (1), z_c' is the intra-bunch coordinate for source particles with retarded location $\theta'=0$, and the infinity limits of z' assume the integrand is negligible at large $|z'|$.

* This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Nuclear Physics under contract DE-AC05-06OR23177.

The retardation requires

$$c(t-t')=R=\left|\vec{R}\right|, \tag{2}$$

where $\vec{R}$ is the vector from S' to S. For $\Delta z=z-z'$, the retardation for case (A) gives [7]

$$\Delta z=(\rho\theta+x')-\beta R, \tag{3}$$

and for case (B), with $\Delta\theta=\theta-\theta'$, it gives [7,8]

$$\Delta z=\rho\left[\Delta\theta-2\beta\left|\sin\left(\Delta\theta/2\right)\right|\right] \tag{4}$$

When $\theta'=0$, we have $\Delta z_c=z-z_c'$ for

$$\Delta z_c=\rho\left[\theta-2\beta\sin\left(\theta/2\right)\right]. \tag{5}$$

Note that for case (A), the electric field on S exerted by S' appears to be originated from S_p' in Fig. 1, which is the location of S' at time t were the source particle S' to continue its straight-path motion at t' with $v=\beta c$ without turning onto the circular orbit as the test particle S does. In the coordinate frame $(\vec{e}_x,\vec{e}_y)$, we have

$$\vec{R}'=(R_x',R_y)=\left[\Delta z-\rho(\theta-\sin\theta),\rho(1-\cos\theta)\right]. \tag{6}$$

The LW field in Eq. (1) can be written as

$$E_{s0}^{(A)}(\theta,t;z')=e\frac{\gamma R_x'\cos\theta+\gamma R_y\sin\theta}{(\gamma^2R_x'^2+R_y^2)^{3/2}} \tag{7}$$

where the dependence on z' and t are contained in Δz of Eq. (6). Note that this longitudinal field is strongest when $R_x'=0$, or when the field lines originated from S_p' shines right on top of S. From the relation between the LW field and the LW scalar and vector potentials $\left(\Phi_0^{(A)},\vec{A}_0^{(A)}\right)$, it has been shown [14]

$$E_{s0}^{(A)}=-\frac{d\Phi_0^{(A)}}{\beta cdt}+\frac{\partial}{\beta c\partial t}\left(\Phi_0^{(A)}-\vec{\beta}\cdot\vec{A}_0^{(A)}\right)=-\frac{\partial V_0^{(A)}}{\partial\Delta z}, \tag{8}$$

with

$$V_0^{(A)}(\theta,\Delta z)=\Lambda_0+\left(\Phi_0^{(A)}-\vec{\beta}\cdot\vec{A}_0^{(A)}\right)$$

for

$$\Lambda_0(\theta,\Delta z)=e\frac{\sin\theta}{R_y}-e\frac{(1-\cos\theta)+R_x'\sin\theta/R_y}{\sqrt{R_x'^2+\gamma^{-2}R_y^2}},$$

which satisfies $\Lambda_0(\theta,\Delta z)=0$ when $\Delta z\to\infty$, and

$$(\Phi_0^{(A)}-\vec{\beta}\cdot\vec{A}_0^{(A)})=e\frac{1-\beta^2\cos\theta}{\sqrt{R_x'^2+\gamma^{-2}R_y^2}}.$$

Correspondingly, for case (B), we have

$$E_{s0}^{(B)}=-\frac{\partial}{\partial\Delta z}\left(\Phi_0^{(B)}-\vec{\beta}\cdot\vec{A}_0^{(B)}\right) \tag{9}$$

for

$$(\Phi_0^{(B)}-\vec{\beta}\cdot\vec{A}_0^{(B)})=e\frac{1-\beta^2\cos\Delta\theta}{\left|2\rho\sin\left(\Delta\theta/2\right)\right|\left[1-\beta\cos\left(\Delta\theta/2\right)\right]},$$

with $\Delta\theta$ implicitly depending on Δz via Eq. (5).

The wakefield on S in Eq. (1) can be obtained by using the LW fields in Eqs. (8) and (9) followed by integration by parts, yielding

$$E_s(\theta,t)=E_s^{(A)}+E_s^{(B)} \tag{10}$$

for

$$E_s^{(A)}=\lambda(z-\Delta z_c(\theta))\Lambda_0(\theta,\Delta z_c(\theta))+\int_{\Delta z(\theta)}^{\infty}V_0^{(A)}(\theta,\Delta z)\frac{\partial\lambda(z-\Delta z)}{\partial\Delta z}d\Delta z \tag{11}$$

and

$$E_s^{(B)}=-e\int_0^{\theta}g(\Delta\theta)\partial_z\lambda\left(z-\left[\rho\Delta\theta-\beta2\rho\sin\left(\Delta\theta/2\right)\right]\right)d\Delta\theta-e\int_0^{\infty}g(\Delta\theta)\partial_z\lambda\left(z+\left[\rho\Delta\theta+\beta2\rho\sin\left(\Delta\theta/2\right)\right]\right)d\Delta\theta \tag{12}$$

with

$$g(\Delta\theta)=\left[\gamma^{-2}+\beta^2(1-\cos\Delta\theta)\right]/\left(2\sin\left(\Delta\theta/2\right)\right).$$

Here for $E_s^{(A)}$ in Eq. (11), we have

$$\Lambda_0(\theta,\Delta z_c(\theta))=-e\tan\left(\theta/4\right)/\rho, \tag{13}$$

and

$$V_0^{(A)}(\theta,\Delta z)=\frac{e}{\rho}\tan^{-1}\left(\frac{\theta}{2}\right)\left(1-\frac{R_x'-\gamma^{-2}R_y\tan^{-1}\theta}{\sqrt{R_x'^2+\gamma^{-2}R_y^2}}\right). \tag{14}$$

The latter becomes a step function at $\gamma\to\infty$

$$V_0^{(A)}(\theta,\Delta z)\Big|_{\gamma\to\infty}\approx\frac{4e}{\rho\theta}H\left(\frac{\rho\theta^3}{6}-\Delta z\right) \tag{15}$$

for the Heaviside function

$$H(x)=1\ (\text{for } x\geq 0),\ H(x)=0\ (\text{for } x<0). \tag{16}$$

At $\gamma\to\infty$ and θ^2 1, Eq. (11) reduces to results in [9]

$$E_s(\theta,t)\approx\frac{4e}{\rho\theta}\left[\lambda\left(z-\frac{\rho\theta^3}{6}\right)-\lambda\left(z-\frac{\rho\theta^3}{24}\right)\right]-\frac{2e}{\left(3\rho^2\right)^{1/3}}\int_{z-\rho\theta^3/24}^{z}\frac{1}{\left(z-z'\right)^{1/3}}\frac{\partial\lambda(z')}{\partial z'}dz' \tag{17}$$

With the above wakefield expressions, we can obtain the CSR impedance by

$$Z(k)=\frac{E_s(\theta,t)}{-e\tilde{\lambda}(k)}e^{-ikz},\quad \text{for } \lambda(z)=\tilde{\lambda}(k)e^{ikz}. \tag{18}$$

For the entrance-CSR, we get from Eqs. (10)-(14)

$$Z(k,\theta)=Z^{(A)}(k,\theta)+Z^{(B)}(k,\theta) \tag{19}$$

with

$$Z^{(A)}=\frac{1}{\rho}\tan\left(\frac{\theta}{4}\right)e^{-ik\Delta z_c(\theta)}+ik\int_{\Delta z_c(\theta)}^{\infty}\left(V_0^{(A)}(\theta,\Delta z)/e\right)\cdot e^{-ik\Delta z}d\Delta z \tag{20}$$

(the first term is negligible) for $V_0^{(A)}(\theta,\Delta z)$ in Eq. (14), and

$$Z^{(B)}=ik\int_0^{\Delta z_c(\theta)}g(\Delta\theta)\cdot e^{-ik\rho\left[\Delta\theta-2\beta\sin(\Delta\theta/2)\right]}d\Delta\theta+ik\int_0^{\infty}g(\Delta\theta)\cdot e^{ik\rho\left[\Delta\theta+2\beta\sin(\Delta\theta/2)\right]}d\Delta\theta \tag{21}$$

ISBN 978-3-95450-182-3

For large θ, Eq. (21) gives the steady-state CSR impedance for finite γ [13], or $Z(k,\theta)\approx Z^{(B)}(k,\theta)$. At $\gamma\to\infty$, using $V_0^{(A)}(\theta,\Delta z)$ in Eq. (15), one gets the ultrarelativistic entrance impedance [12,15]

$$Z(k,\theta)=\frac{4}{\rho\theta}\left[e^{-i\mu}-e^{-4i\mu}+(i\mu)^{1/3}f(\mu)\right] \qquad (22)$$

for $\mu=k\rho\theta^3/24$, and $f(\mu)=\Gamma(2/3)-\Gamma(2/3,i\mu)$.

EXIT CASE

The exit case is illustrated in Fig. 2 [7].

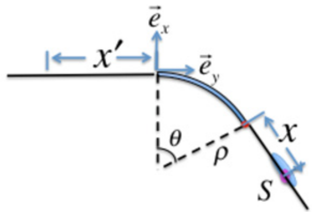

Figure 2: Geometry of interaction on S for exit case.

The longitudinal wakefield on S is given by

$$E_s(x,t)=E_s^{(C)}+E_s^{(D)}+E_s^{(E)}$$

with

$$E_s^{(C)}=\int_{-\infty}^{z'(\theta)}E_{s0}^{(C)}(x,t;z')\lambda(z')dz',$$

$$E_s^{(D)}=\int_{z'(\theta)}^{z'(0)}E_{s0}^{(D)}(x,t;z')\lambda(z')dz', \qquad (23)$$

$$E_s^{(E)}=\int_{z'(0)}^{\infty}E_{s0}^{(E)}(x,t;z')\lambda(z')dz',$$

for integration limit $z'(\theta')$ given by Eq. (24). Here $E_{s0}^{(C)}(x,t;z')$ is the LW field on S at (x,t) generated from S' at (x',t') upstream of the dipole, $E_{s0}^{(D)}(x,t;z')$ is from S' at (θ',t') inside the dipole, and $E_{s0}^{(E)}(x,t;z')$ is from S' on the same straight path as S downstream of the dipole. Note that $E_{s0}^{(C)}(x,t;z')$ can be expressed exactly as $E_{s0}^{(A)}(x,t;z')$ in Eqs. (7)-(9), with R_x' and R_y replaced by

$$R_x'=\Delta z-\rho(\theta-\sin\theta)-x(1-\cos\theta),$$

$$R_y=\rho(1-\cos\theta)+x\sin\theta.$$

The focus of our discussion will be on the $E_s^{(D)}$ term in Eq. (23), or case (D) in Ref. [7]. Here θ' and z' are related by retardation relation

$$\Delta z=x+\rho\theta'-\beta R(x,\theta') \qquad (24)$$

for

$$R(x,\theta')=\sqrt{x^2+2\rho x\sin\theta'+\left[2\rho\sin(\theta'/2)\right]^2}.$$

With Taylor expansion for $\gamma\gg1$ and $\theta'^2\ll1$, it is shown [7]

$$E_{s0}^{(D)}(x,t;z')dz'=-\frac{\partial V_0^{(D)}(x,\theta')}{\partial\theta'}d\theta'$$

for

$$V_0^{(D)}=\frac{4e}{\rho}\frac{2\gamma^{-2}(\theta'+x/\rho)+\theta'^2(\theta'+2x/\rho)}{\gamma^{-2}\left[2(\theta'+x/\rho)\right]^2+\theta'^2(\theta'+2x/\rho)^2} \qquad (25)$$

Therefore

$$E_s^{(D)}=-V_0^{(D)}(x,\theta')\lambda\left(z-\Delta z(x,\theta')\right)\Big|_0^{\theta}+\int_0^{\theta}V_0^{(D)}(x,\theta')\frac{\partial}{\partial\theta'}\lambda(z-\Delta z(x,\theta'))d\theta' \qquad (26)$$

At $\gamma\to\infty$, we have

$$V_0^{(D)}(x,\theta')=\frac{4e}{\rho}\frac{1}{\theta'+2x/\rho} \qquad (27)$$

and $E_s^{(D)}$ in Eq. (26) reduces to the well-known expression for exit-CSR impedance for ultrarelativistic beams [9] (with W=-E_s/e)

$$E_s^{(D)}=-\frac{4e}{\rho}\left(\frac{\lambda(z-\Delta z(x,\theta'))}{\theta'(\Delta z)+2x/\rho}\Bigg|_{\Delta z_{\min}}^{\Delta z_{\max}}+\int_{z-\Delta z_{\max}}^{z-\Delta z_{\min}}\frac{\partial_{z'}\lambda(z')/\partial z'}{\theta'(z-z')+2x/\rho}dz'\right)$$

where $\theta'(\Delta z)$ is given in Eq. (26), with $\theta'(\Delta z_{\min})=0$ and $\theta'(\Delta z_{\max})=\theta$.

Subsequently, the exit-CSR impedance can be derived using Eq. (18), yielding

$$Z^{(D)}(k,x)=\int_0^{\theta}\frac{\partial V_0^{(D)}(x,\theta')}{\partial\theta'}e^{-ik\Delta z(x,\theta')}d\theta' \qquad (28)$$

for $V_0^{(D)}(x,\theta')$ in Eq. (25), or

$$Z^{(D)}(k,x)=V_0^{(D)}(x,\theta')e^{-ik\Delta z(x,\theta')}\Big|_0^{\theta}-\int_0^{\theta}V_0^{(D)}(x,\theta')\left(\frac{\partial}{\partial\theta'}e^{-ik\Delta z(x,\theta')}\right)d\theta' \qquad (29)$$

The above expression applies for large but finite γ, and it uses the explicit retardation relation $\Delta z(x,\theta')$ in Eq. (24) without the need to solve its inverse function. At high frequency, we found Eq. (28) is less prone to numerical error as compared to Eq. (29).

CONCLUSION

In this paper, we present the analytical expression of CSR impedance for the entrance and exit problems for finite γ, that reduces to the well-known results for $\gamma\to\infty$. Edge term contributions, singularity removal by renormalization [7], and behaviours of the wakefield and impedance in various parameter regimes will be discussed elsewhere [16].

ACKNOWLEDGEMENT

We thank Chris Tennant and Steve Benson for helpful discussions.

REFERENCES

[1] M. Borland, *Phys. Rev. ST Accel. Beams* **11**, 030701 (2008).

[2] I. Pogorelov *et al.*, in *Proc. PAC'07*, 1165 (2007).

[3] Heifets, Stupakov, and Krinsky, *Phys. Rev. ST Accel. Beams* **5**, 064401 (2002) and its erratum paper, *Phys. Rev. ST Accel. Beams* **5**, 129902 (2002).

[4] Huang and Kim, *Phys. Rev. ST Accel. Beams* **5**, 074401 (2002) and its erratum paper, *Phys. Rev. ST Accel. Beams* **5**, 129903 (2002).

[5] C-Y. Tsai *et al.*, *Phys. Rev. ST Accel. Beams* **19**, 114401 (2016).

[6] C-Y. Tsai and R. Li, JLAB-TN-15-019 (2015).

[7] E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, *Nucl. Instrum. Methods Res. Sect.* A 398, 373 (1997).

[8] Ya. S. Derbenev, J. Rossbach, E. L. Saldin, and V. D. Shiltsev, TESLA-FEL-Report 1995-05 (1995).

[9] Stupakov and P. Emma, SLAC Technical Report, LCLS-TN-01-12 (2001).

[10] D. Sagan *et al.*, *Phys. Rev. ST Accel. Beams* **12**, 040703 (2009).

[11] C. Mayes and G. Hoffstaetter, *Phys. Rev. ST Accel. Beams* **12**, 024401 (2009).

[12] C. Mitchell and J. Qiang, in *Proc. IPAC'13*, 1832 (2013).

[13] R. Li and C-Y. Tsai, in *Proc. IPAC'15*, 709 (2015).

[14] R. Li, C. L. Bohn, and J. J. Bisognano, *Proc. SPIE*, vol. 3154, p.223, San Diego (1997).

[15] Here there is a sign difference from Ref. [12] in the impedance definition.

[16] R. Li and C-Y. Tsai, presented at IPAC'17, Copenhagen, Denmark, paper WEPIK113, this conference.

ISBN 978-3-95450-182-3