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GAIN OF HARD X-RAY FEL AT 3GeV AND REQUIRED PARAMETERS

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Abstract

We develop a tool for calculation to study the conditions for a hard x-ray FEL oscillator at 3 GeV. We show that the approach developed by K.J. Kim, *et.al.* [1–3] for the small signal low gain formula can be modified so that the gain can be derived wihout taking the "no focusing approximation" adopted in the approach, so that a strong focusing can be applied. We also derive the formula to allow for gain calculation of harmonic lasing. We show that the relation of gain with harmonic number is not sufficient to determine whether harmonic lasing is favorable, also the relation between harmonic number, the undulator field strength, gap, period and energy spread must be analyzed together.

INTRODUCTION

An x-ray FEL oscillator based on transverse gradient undulator (TGU) considered by APS and SLAC collaboration [1, 2] provides a promising direction for storage ring based fully coherent hard x-ray source. The difficulty associated with the relative large energy spread of 10^{-3} in storage ring is mitigated by introducing TGU and vertcal dispersion in the FEL by a trade-off with increased beam size.

At 3 GeV the resonance condition requires us to consider harmonic lasing as in [4, 5]. A crucial point in considering the harmonic lasing is, we emphasize, that the relation of gain with harmonic number is not sufficient to determine whether harmonic lasing is favorable, also the relation between harmonic number, the undulator field strength, gap, period and energy spread must be analyzed together. We adop the approach in [1, 2] for the low gain formula which is based on the low gain formula derived by K.J. Kim [3]. For this purpose we need to follow through the derivation to explicitly allow for harmonic lasing.

Before going into more detailed analysis, we first consider gain formula in 1D Madey theorem with harmonic number h and when energy spread is negligible the gain can be cast in a form convenient for scaling the gain with harmonic number h and undulator period λ_u:

$$G_{1D} = \left(\frac{hK^2[JJ]_h^2}{\lambda_u}\right)\left(\pi^2\frac{I}{I_A}\frac{L^3}{\gamma^3\Sigma}\right)\left(\frac{d}{d\Phi}\left(\frac{\sin\Phi}{\Phi}\right)^2\right) \quad (1)$$

where K is the undulator parameter given by the peak field B_{peak} in the resonance condition

$$\lambda_s = \frac{\lambda_u}{2\gamma_0^2 h}(1+\frac{K^2}{2}) \quad (2)$$

γ_0 is the resonant electron beam energy in unit of electron rest mass, λ_s is the FEL wavelength, $L = N_u\lambda_u$ is the undulator length with number of period N_u. $[JJ]_h = J_{(h-1)/2}\left(\frac{hK^2}{4+2K^2}\right) - J_{(h+1)/2}\left(\frac{hK^2}{4+2K^2}\right)$ is the

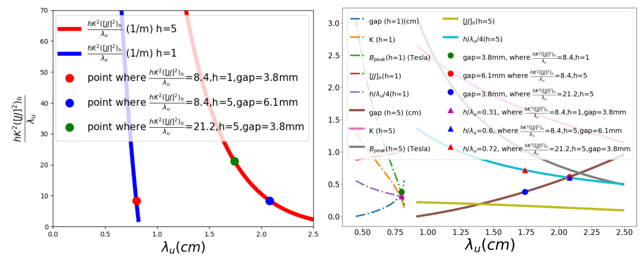

Figure 1: a: $\frac{K^2[JJ]_n^2 h}{\lambda_u}$ vs. λ_u, compare h = 5 with h = 1 b: K, Bw, [JJ], gap and h/λ_u for h = 5 and h = 1.

Bessel factor, $\Sigma = 2\pi\sigma_x\sigma_y$ is the electron beam cross section area with σ_x, σ_y the RMS beam size, I is beam peak current. $I_A = 4\pi mc^3\epsilon_0/e \approx 17$ kA is the Alfven current. $\Phi = \pi\Delta\nu N_u - 2\eta h\pi N_u$ is the phase advance in the undulator due to detuning, with $\eta = (\gamma-\gamma_0)/\gamma_0$ is the relative energy detuning of mean energy γ from resonance, $\Delta\nu = h(\omega-\omega_s)/\omega_s$ is the laser frequency detuning from resonance frequency $\omega_s = 2\pi/\lambda_s$ with harmonic number h. The effect of the energy spread can be obtained by an average of the 3rd factor $\frac{d}{d\Phi}\left(\frac{\sin\Phi}{\Phi}\right)^2$ over the energy spread σ_η, which gives the effect of the spread of Φ.

For a given λ_s, γ, peak current I, undulator length L_u, and Σ, we consider the scaling relation of the gain and h, λ_u given by Eq. (1) in the first factor $\left(\frac{K^2[JJ]_h^2 h}{\lambda_u}\right)$. The second factor is fixed by these parameters. However, we must also take into accout the K.Halbach formula [6] about the relation between the undulator gap g, λ_u and K:

$$K = \frac{e\lambda_u B_{peak}}{2\pi mc} = 93.43\lambda_u B_{peak}$$
$$B_{peak} = 3.1\exp\left(-5.47\frac{g}{\lambda_u} + 1.8\left(\frac{g}{\lambda_u}\right)^2\right) \quad (3)$$

For 3 Gev beam, and for $\lambda_s = 0.12$ nm, we plot $\frac{K^2[JJ]_h^2 h}{\lambda_u}$ as function of λ_u for $h = 1.5$ respectively in Fig. 1a. In Fig. 1b, we plot the parameters K, $[JJ]_h$, B_{peak},the gap g and h/λ_u From Fig. 1b we see from the point of view of gap alone, we would need to consider possibility for higher harmonic number. As pointed out in [1, 7, 8] with high reflectivity of X-ray mirror, the gain should be around 20 to 30%. The comparison of two pairs of points with same gap or same first factor shows that for the contribution to the first factor, for the same gap, higher harmonic number has higher gain, while for the same gain, higher harmonic number has larger gap.

However, the contribution from the 3rd factor $\frac{d}{d\Phi}\left(\frac{\sin\Phi}{\Phi}\right)^2$ is more complicated, with maximum of 0.54 at $\Phi = \pi\Delta\nu N_u - 2\eta h\pi N_u = -1.3$, the term

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$2\eta h\pi N_u = 2\eta\pi L_u \frac{h}{\lambda_u}$ in Φ has a spread propotional to h/λ_u due to the energy spread σ_η, the increased spread would reduce the average value of the 3rd factor. Fig. 1b shows h/λ_u also affects the gain when the energy spread is not negligible.

In the formulation developed in [1–3] about the 3D gain, if we assume the gradient is in vertical direction, then the undulator parameter $K = K_0(1+\alpha y)$, the energy is $\gamma = \gamma_0(1+\eta)$, for $\alpha y \ll 1, \eta \ll 1$, the resonance condition becomes

$$\lambda = \frac{\lambda_u}{2\gamma^2 h}(1+\frac{K^2}{2}) \quad (4)$$

$$= \frac{\lambda_u}{2\gamma_0^2 h}\frac{1+\frac{K_0^2}{2}(1+\alpha y)^2}{(1+\eta)^2}$$

$$\approx \frac{\lambda_u}{2\gamma_0^2 h}\left(1+\frac{K_0^2}{2}\right)\left(1+\frac{2K_0^2}{2+K_0^2}\alpha y - 2\eta\right) \quad (5)$$

If we assume the dispersion is D in vertical direction, the vertical distribution is in gaussian form $\exp(-\frac{(y-D\eta)^2}{2\sigma_y^2})$. the centroid of the electron beam with energy η is shifted to $D\eta$, then for very small σ_y, we can take $y \approx D\eta$, and the deviation from resonance condition due to the spread in η is given by $\frac{2K_0^2}{2+K_0^2}\alpha D\eta - 2\eta$. If $\alpha D = \frac{2+K_0^2}{K_0^2}$, then the gain reduction due to large energy spread is mitigated, in the optimization we also allow α to deviate from this condition.

Clearly, to study the beam quality required for an x-ray FEL working in the medium energy range, we need to apply the gain formula developed in [1–3] for optimization, with the 3D effect of diffraction, beam divergence and betatron motion taken into account, and in particular, with an addition to include harmonic lasing. In following the procedure to optimize the parameters, we realized that in the formulation in [1–3], a "no focusing" approximation is taken in the process of derivation of the gain formula using the "brightness function", as explained clearly after the Eq. (24) of [3] by K.J. Kim. The "no focusing" approximation essentially neglects the focusing. In our optimization process, we often found that we need to increase the focusing in the undulator, and we reached a set of parameters which violated the condition required by the "no focusing" approximation.

We derived a gain formula without taking the "no focusing" approximation and without using the "brightness function". Due to space limit we only give the result here. We first followed [3] to describe the general 3D gain formula resemble 1-D Madey theorem, the Eq. (23) in [3]. This general gain formula, without being given a specific gaussian form of the electron beam distribution and the input radiation field, is our starting point, which is also the last step in [1–3] right before taking the "no focusing" approximation. Then we carried out a multivariable gaussian integration without taking the "no focusing" approximation and reduce the gain formula to a double integral, similar to the result of [1–3]. In Section III we present an examples to see the required parameters for an X-ray FEL oscillator in a 3 GeV energy storage ring.

GAIN FORMULA

The gain in small signal, low gain regime taking into account of the 3D effect of diffracrion, beam divergence and betatron motion, is given in the following Eq. (6), which is the Eq. (23) of [3], with a minor elaboration of introducing the transverse gradient and dispersion for TGU as in [1, 2]. For convenience we adopt nearly identical notation as [3], with only few exceptions to unite with the notations in [1–3] and the notations we used in the early development of the coupled Maxwell-Vlasov equations for high gain FEL [9–12]. The gain expression has multiple integrations to be carried out for application:

$$G = -c_h \int d\eta dx dp \overline{F}(\mathrm{x},\mathrm{p},\eta;0)\times \frac{\frac{\partial}{\partial\eta} \mid \int d\phi A_\nu^{(0)}(\phi,0)U_\nu^*(\eta,\phi,\mathrm{p},\mathrm{x}) \mid^2}{\int \mid A^{(0)}(\phi) \mid^2 d\phi} \quad (6)$$

where $\overline{F}$ is the electron distribution function, U_ν^*, is the amplitude of the undulator radiation, and $A^{(0)}$ is the amplitude of the incoming laser. Due space limit the notation is referred to [1–3]. About "no focusing" approximation, the condition for neglecting $k_{\beta x}^2 x_0^2$ in $(p_{0x}-\phi_x)^2 + k_{\beta x}^2 x_0^2$ in U_ν^* , because p_x and $k_{\beta x}x$ are about the order of $k_{\beta x}\sigma_x$, corresponds to require $\sigma_{\phi x} \gg k_{\beta x}\sigma_x$ ($\sigma_{\phi x}$ is radiation angular spread, $k_{\beta x}\sigma_x$ is beam angular spread). Same way it requires $\sigma_{\phi y} \gg k_{\beta y}\sigma_y$. We found often our optimization leads to a set of parameters which violate this condition, in particular, when emittance is is not very small and we need to increase $k_\beta = 2\pi/\beta$. For this purpose, we kept the term $k_{\beta x}^2 x_0^2$, and realized that with Eq.(6) as our starting point, it is still possible to carried out multiple gaussian intergration.

The result is

$$G = -\left(\frac{hK[JJ]_h^2}{\lambda_u}\right)\left(\frac{I}{I_A}\frac{\pi^2}{\gamma^3}\right)(2I_{3D})\frac{2\pi w_x w_y}{\lambda^2},$$

$$I_{3D} = \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} dsdz \frac{-i(s-z)\exp(i\Delta\nu k_u(s-z))}{\sqrt{(D_xD_{px}-\sigma_x^2\sigma_{px}^2B_{xp}^2)}\sqrt{D_{py}}\sqrt{D_{\eta y}}} \quad (7)$$

$$\times \exp\left(\frac{N_{\eta y}}{D_{\eta y}}\sigma_\eta^2\sigma_y^2\right)\frac{1}{D_R(s,z)}$$

where $D_R(s,z), D_x, D_{px}, D_{py}, B_{xp}$ are given in the following Eqs. (8) and (9).

$$D_R(s,z) \equiv \sqrt{z_{Rx}-is}\sqrt{z_{Ry}-is}\sqrt{z_{Rx}+iz}\sqrt{z_{Ry}+iz} \quad (8)$$

where $z_{Rx} = \frac{\pi w_x^2}{\lambda}$ is Rayleigh range, with $w_x = 2\sigma_{rx}$ the guassian beam waist of radiation (similar for y direction)

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$$D_x = 1 - 2\sigma_x^2\left[c_x(s) + c_x^*(z)\right] - ikk_{\beta x}^2\sigma_x^2(s-z)$$
$$D_{px} = 1 - 2\sigma_{px}^2\left[c_{px}(s) + c_{px}^*(z)\right] - ik\sigma_{px}^2(s-z)$$
$$D_{py} = 1 - 2\sigma_{py}^2\left[c_{py}(s) + c_{py}^*(z)\right] - ik\sigma_{py}^2(s-z)$$
$$B_{xp} = -2\left[c_{xp}(s) + c_{xp}^*(z)\right]$$
$$c_x(s) \equiv -k\frac{\cos^2(\frac{s}{\beta_x})}{2(z_{Rx} - is)}, c_{px}(s) \equiv -k\frac{\beta_x^2\sin^2(\frac{s}{\beta_x})}{2(z_{Rx} - is)},$$
$$c_{xp}(s) \equiv -k\frac{\beta_x\cos(\frac{s}{\beta_x})\sin(\frac{s}{\beta_x})}{2(z_{Rx} - is)}$$
$$c_y(s) \equiv -k\frac{\cos^2(\frac{s}{\beta_y})}{2(z_{Ry} - is)}, c_{py}(s) \equiv -k\frac{\beta_y^2\sin^2(\frac{s}{\beta_y})}{2(z_{Ry} - is)}$$
$$D_{\eta y} = 1 + \left(\sigma_y^2 + D^2\sigma_\eta^2\right)\left(2A_y - \frac{1}{\sigma_y^2} - \frac{\sigma_{py}^2 B_{yp}^2}{D_{py}}\right)$$
$$N_{\eta y} = C_{ss}c_s^2(s,z) + C_0 + C_s c_s(s,z) + C_{sc}c_s(s,z)c_c(s,z) \quad (9)$$
$$+ C_{cc}c_c^2(s,z) + C_c c_c(s,z)$$

where $c_s(s,z) \equiv \left(\sin(\frac{s}{\beta_y}) - \sin(\frac{z}{\beta_y})\right)$,

$$c_c(s,z) \equiv \left(\cos(\frac{s}{\beta_y}) - \cos(\frac{z}{\beta_y})\right)$$

Their coefficients in $N_{\eta y}(s,z)$ are also functions of s, z:

$$C_{ss} = B_\alpha^2 A_\eta, C_0 = B_\eta^2\left(A_y - \frac{B_{yp}^2}{4A_{py}}\right), C_s = B_\eta B_\alpha B_{\eta y}$$
$$C_{sc} = \frac{B_{yp}B_\alpha^2\beta_y A_\eta}{A_{py}}, C_{cc} = \frac{B_\alpha^2\beta_y^2}{A_{py}}\left(A_\eta A_y - \frac{1}{4}B_{\eta y}^2\right),$$
$$C_c = \frac{B_\eta B_\alpha B_{\eta y}B_{yp}\beta_y}{2A_{py}}$$
$$A_y \equiv \frac{1}{2\sigma_y^2} - \left[c_y(s) + c_y^*(z)\right] - i\frac{k}{2}k_{\beta y}^2(s-z), \quad (10)$$
$$A_{py} \equiv \frac{1}{2\sigma_{py}^2}D_{py}, A_\eta \equiv \frac{1}{2\sigma_\eta^2} + \frac{D^2}{2\sigma_y^2}$$
$$B_\alpha \equiv i\nu\frac{2K_0^2}{2+K_0^2}\alpha k_u\beta_y, B_{yp} = -2\left[c_{yp}(s) + c_{yp}^*(z)\right],$$
$$B_\eta = 2i\nu k_u(s-z), B_{\eta y} = -\frac{D}{\sigma_y^2}$$

EXAMPLES OF GAIN CALCULATION

The Eq. (7) is the main result of this paper. Our goal is to apply this formula to explore the possibility of a hard x-ray FEL for a light source at 3 GeV, and in particular, to find the required electron beam quality and undulator for an upgrade of NSLSII to drive an hard x-ray FEL to help to study whether it is possible. Confirmation with simulation is still work in progress.

First we assume a 3 GeV FEL at 0.12 nm, the peak current $I = 280$ A, and bunch length 50 ps. The revolution period is about 2.6 µs, we approximate the bunch as a flat top pulse, the bunch current is 5.3 mA. As discussed in the Section I, we plot the first factor $\frac{K^2[JJ]_h^2 h}{\lambda_u}$ in the 1D gain formula as function of λ_u in Fig. 1a, and plot the gap g vs. λ_u in Fig. 1b. For $h = 5$, as a compromise between larger gain and gap, we choose $\lambda_u = 2.26$ cm and $\frac{K^2[JJ]_h^2 h}{\lambda_u} = 4.96\,\mathrm{m}^{-1}$, with gap $g = 7.5$ mm, $K_0 = 1.29$.

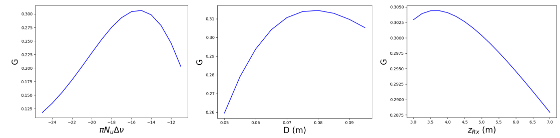

Figure 2: G vs. a: $\Delta\nu$, b:D, c: z_{Rx}

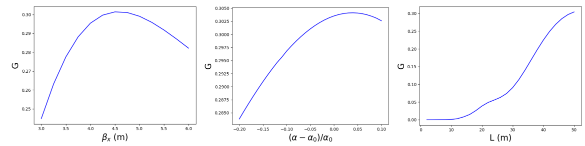

Figure 3: G vs. a:β_x, b:$\frac{\alpha-\alpha_0}{\alpha_0}$, c:$L_u$

We assume the emittance $\epsilon_x = 25$ pm, $\epsilon_y = 5$ pm energy spread $\sigma_\eta = 10^{-3}$, and ondulator lengh $L = 50$ m, and scan the 6 variables: detuning $\Delta\nu$, horizontal dispersion D, focusing beta functions β_x, β_y, the input radiation Rayleigh ranges z_{Rx}, z_{Ry}, and the transverse gradient α to find maximum gain. During the scan we find the transverse gradient α should be allowed to deviate from α_0 (where $\alpha_0 D = \frac{2+K_0^2}{K_0^2}$). Actually we find the optimized α close but larger than α_0.

The plot of scan in the last cycle is given in Figs. 2 and 3. The paramters for maximum G are given in Table 1. For all the cases in Table I the energy spread is $\sigma_\eta = 10^{-3}$, the undulator length is taken as $L = 50$ m, the x-ray wavelength is $\lambda_s = 0.12$ nm, gap $g = 7.5$ mm. Dispersion D is vertical.

Table 1: Parameters for Maximum G at 3 GeV

I (A)	λ_u	h	G(%)	K_0
280	2.26	5	30.4	1.29
ϵ_x (pm)	ϵ_y (pm)	D (cm)	β_x (m)	β_y (m)
25	5	6.5 (V)	4.5	5.5
z_{Rx} (m)	z_{Ry} (m)	$\alpha\,(\mathrm{m}^{-1})$	σ_x(µm)	σ_y(µm)
4	5.5	35.2	10	5.2

SUMMARY

We developed a gain formula for a hard x-ray FEL at 3 GeV so that we do not need to take "no focusing approximation" in the calculation, in hope this can be of use in exploring the possibility for x-ray FEL in this energy range. The formula allow gain calculation with harmonic lasing. An example indicates hard x-ray FEL at 3 GeV seems possible, even though it sets rather challenging conditions.

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