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3-D QUANTUM LIFETIME

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Abstract

The quantum lifetime of electron beam in storage rings is defined by the particle loss that caused by the aperture limitation. Based on the equilibrium beam distribution produced by radiation damping and quantum excitation, the 1-d quantum lifetime has been well studied by A. Piwinski. In this paper, we give the derivation of the 3-d quantum lifetime, which can be applied to the machines with elliptical aperture and momentum acceptance.

INTRODUCTION

Except photon sources, the high-intensity high-current electron storage ring is also needed for electron cooling devices, electron-ion colliders and other applications [1,2]. In such machines, the transverse dynamic aperture and momentum acceptance may be small since the high intensity and the use of insertion devices for strong synchrotron radiation. Even though the Touschek effect is the main reason for the particle loss in electron storage rings, the quantum lifetime is still an important factor that needs to be considered in the machine design and operation.

Since the aperture of an accelerator is limited by accelerator components such as vacuum chambers, beam position monitors, etc., the electron beam with Guassian distribution, which has an infinitely long tail, can get lost at the aperture, which defines the quantum lifetime. In addition, dynamic aperture is also a reason for the particle loss. Normally, the physical and dynamic apertures are so large that the quantum lifetime is smaller than the damping time. So, the quantum lifetime can be calculated by the flux of electron passing through the aperture based on the equilibrium distribution of the beam, which is determined by the quantum excitation, radiation damping and other effects. The 1-d quantum lifetime that only considers one dimensional aperture has been well modeled by A. Piwinski [3]. In this paper, we give the derivation of the 3-d quantum lifetime, in which the ellipsoidal aperture is considered. The simulation is consistent with the formula both for 1-d and 3-d lifetime model.

BEAM DISTRIBUTION

Assuming there is no external electromagnetic forces in the system, the Fokker-Plank equation of the electron beam distribution is:

$$\frac{\partial w}{\partial t} + \frac{\partial I_x}{\partial \epsilon_x} + \frac{\partial I_y}{\partial \epsilon_y} + \frac{\partial I_p}{\epsilon_p} = 0, \tag{1}$$

where $w = w(\epsilon_x, \epsilon_y, \epsilon_z, t)$ is the beam distribution function ($\int wdw = 1$), and I is the flux of density, which is defined by:

$$I_{x,y,z} = 2\lambda_{x,y,z}\epsilon_{x,y,z}M_{x,y,z} + \epsilon_{x,y,z}M_{x,y,z}\frac{\partial w}{\partial \epsilon_{x,y,z}}, \tag{2}$$

where λ is the damping rate and M is the factor of quantum excitation. Considering Gaussian distribution, the stationary beam distribution function ($\partial w/\partial t = 0$) is

$$w(\epsilon_x, \epsilon_y, \epsilon_z) = \frac{1}{8\hat{\epsilon}_x\hat{\epsilon}_y\hat{\epsilon}_z}exp(-\frac{\epsilon_x}{2\hat{\epsilon}_x} - \frac{\epsilon_y}{2\hat{\epsilon}_y} - \frac{\epsilon_z}{2\hat{\epsilon}_z}), \tag{3}$$

where $\hat{\epsilon}_x$, $\hat{\epsilon}_y$ and $\hat{\epsilon}_z$ are the rms beam emittance, which are defined by $\hat{\epsilon}_{x,y,z} = M_{x,y,z}/4\lambda_{x,y,z}$. It is worth noting that the longitudinal emittance here corresponds to the momentum spread $\epsilon_z = \delta_p^2$.

BEAM LIFETIME

We assume the aperture of the system is large (>3σ) so that the Gaussian distribution of electron beam can be kept at the equalibrium state. Then the beam lifetime can be defined by the flux of electron passes inward or outward through the aperture:

$$\begin{aligned}\frac{1}{\tau} &= -\frac{1}{N}\frac{dN}{dt} = -\frac{1}{dt}\iiint_V wd\epsilon_x d\epsilon_y d\epsilon_z \\ &= \iiint_V \frac{\partial I_x}{\partial \epsilon_x} + \frac{\partial I_y}{\partial \epsilon_y} + \frac{\partial I_z}{\partial \epsilon_z} d\epsilon_x d\epsilon_y d\epsilon_z \\ &= \frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_z},\end{aligned} \tag{4}$$

where $V = f(\epsilon_x, \epsilon_y, \epsilon_z)$ is the beam acceptance defined by the 3-d aperture. Here we assume the aperture is in the shape of ellipsoid:

$$\frac{\epsilon_x}{A_x^2\hat{\epsilon}_x} + \frac{\epsilon_y}{A_y^2\hat{\epsilon}_y} + \frac{\epsilon_z}{A_z^2\hat{\epsilon}_z} = 1, \tag{5}$$

where $A_x = a/\hat{\sigma}_x$, $A_y = b/\hat{\sigma}_y$ and $A_z = c/\hat{\delta}_p$ are the ratios between the aperture (a, b, c) and the rms size of beam in three dimensions. The lifetime, for example τ_x, can be calculated by:

$$\frac{1}{\tau_x} = \int_0^{A_y^2\hat{\epsilon}_y}\int_0^{A_z^2\hat{\epsilon}_z(1-\frac{\epsilon_y}{A_y^2\hat{\epsilon}_y})} I_{xm}(\epsilon_y, \epsilon_z)d\epsilon_z d\epsilon_y. \tag{6}$$

Based on the method in [3], mutiplying with $exp(2\lambda\epsilon_x/M)/\epsilon_x$ in Eq. (2) and integrate both sides, the maximum flux $I_{xm} = \int_0^{\epsilon_{xm}} \partial I_x/\partial \epsilon_x d\epsilon_x$ can be derived by:

$$\frac{I_{xm}}{\epsilon_{xm}}\int_{-\infty}^{\epsilon_{xm}} exp(\frac{\epsilon_x}{2\hat{\epsilon}_x})d\epsilon_x = M_x w(\epsilon_x = 0, \epsilon_y, \epsilon_z), \tag{7}$$

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then we get:

$$I_{xm} = 2\lambda_x e^{-\frac{\epsilon_{xm}}{2\hat{\epsilon}_x}} w(\epsilon_x = 0, \epsilon_y, \epsilon_z). \quad (8)$$

where $\epsilon_{xm} = A_x^2 \hat{\epsilon}_x (1 - \frac{\epsilon_y}{A_y^2 \hat{\epsilon}_y} - \frac{\epsilon_z}{A_z^2 \hat{\epsilon}_z})$. Then substituing Eq. (8) into Eq. (6), τ_x can be calculated. The calculation in the other two planes are the same. With the help of MATHEMATICA, we obtain

$$\begin{aligned}
\frac{1}{\tau_x} &= \lambda_x A_x^2 A_y^2 A_z^2 (l_{xx} e^{\frac{-A_x^2}{2}} + l_{xy} e^{\frac{-A_y^2}{2}} + l_{xz} e^{\frac{-A_z^2}{2}}) \\
\frac{1}{\tau_y} &= \lambda_y A_x^2 A_y^2 A_z^2 (l_{yx} e^{\frac{-A_x^2}{2}} + l_{yy} e^{\frac{-A_y^2}{2}} + l_{yz} e^{\frac{-A_z^2}{2}}) \quad (9) \\
\frac{1}{\tau_z} &= \lambda_z A_x^2 A_y^2 A_z^2 (l_{zx} e^{\frac{-A_x^2}{2}} + l_{zy} e^{\frac{-A_y^2}{2}} + l_{zz} e^{\frac{-A_z^2}{2}}),
\end{aligned}$$

where

$$\begin{aligned}
l_{xx} &= \frac{A_x^2(4 + A_x^2 - A_y^2 - A_z^2) + A_y^2 A_z^2 - 2(A_y^2 + A_z^2)}{(A_y^2 - A_x^2)^2 (A_z^2 - A_x^2)^2} \\
l_{yy} &= \frac{A_y^2(4 + A_y^2 - A_x^2 - A_z^2) + A_x^2 A_z^2 - 2(A_x^2 + A_z^2)}{(A_y^2 - A_x^2)^2 (A_z^2 - A_x^2)^2} \\
l_{zz} &= \frac{A_z^2(4 + A_z^2 - A_x^2 - A_y^2) + A_x^2 A_y^2 - 2(A_x^2 + A_y^2)}{(A_x^2 - A_z^2)^2 (A_y^2 - A_z^2)^2} \\
l_{xy} &= \frac{-2}{(A_y^2 - A_z^2)(A_y^2 - A_x^2)^2} \\
l_{yx} &= \frac{-2}{(A_x^2 - A_z^2)(A_x^2 - A_y^2)^2} \\
l_{xz} &= \frac{-2}{(A_z^2 - A_y^2)(A_z^2 - A_x^2)^2} \\
l_{zx} &= \frac{-2}{(A_x^2 - A_y^2)(A_x^2 - A_z^2)^2} \\
l_{yz} &= \frac{-2}{(A_z^2 - A_x^2)(A_z^2 - A_y^2)^2} \\
l_{zy} &= \frac{-2}{(A_y^2 - A_x^2)(A_y^2 - A_z^2)^2}.
\end{aligned}$$

Finally, the quantum lifetime is simplified as:

$$\frac{1}{\tau} = A_x^2 A_y^2 A_z^2 (k_x e^{\frac{-A_x^2}{2}} + k_y e^{\frac{-A_y^2}{2}} + k_z e^{\frac{-A_z^2}{2}}), \quad (10)$$

where

$$\begin{aligned}
k_{x,y,z} = &\frac{1}{(A_{z,x,y}^2 - A_{x,y,z}^2)^2 (A_{y,z,x}^2 - A_{x,y,z}^2)^2} \{\lambda_{x,y,z}[\\
& A_{x,y,z}^2 (A_{x,y,z}^2 - A_{y,z,x}^2 - A_{z,x,y}^2) + A_{y,z,x}^2 A_{z,x,y}^2] \\
& + 2A_{x,y,z}^2 (2\lambda_{x,y,z} - \lambda_{y,z,x} - \lambda_{z,x,y}) \\
& + 2A_{y,z,x}^2 (\lambda_{z,x,y} - \lambda_{x,y,z}) \\
& + 2A_{z,x,y}^2 (\lambda_{y,z,x} - \lambda_{x,y,z})\}.
\end{aligned}$$

We see that the quantum lifetime only depends on damping rate λ and the aperture A. If the aperture in the other two dimensions are very large, the formula agrees with the 1-d form: $1/\tau = \lambda A^2 exp(-\frac{A^2}{2})$, which is consistent with [3].

SIMULATION

In this section, we introduce a simple simulation to track the particle loss that due to radiation damping and quantum excitation. In the simulation, the stochastic process is independently calculated in each dimension, in which the dispersion and beam rotation in phase space are not considered. The kinetic equation on each particle, for example in transverse plane, is described by

$$\begin{aligned}
\Delta x &= -\lambda_x x + \sqrt{2\lambda_x \sigma_x^2}\eta \\
\Delta p_x &= -\lambda_x p_x + \sqrt{2\lambda_x \sigma_{p_x}^2}\zeta,
\end{aligned} \quad (11)$$

where λ is the radiation damping rate, σ_x and σ_{px} are rms beam sizes in phase space at the equilibrium state, and η and ζ are random numbers with standard gaussian distribution. Based on the aperture size, the beam lifetime can be calculated by the tracking of particle loss.

Firstly, we finished the 1-d simulation and compared the results with the theoretical model. We start with the beam that is already in the equilibrium state, then track the particle loss at the aperture during the stochastic process. The damping time is $1/\lambda$=10 ms and the aperture is set to be 4.6σ.

Figure 1 shows the evolution of the beam emittance and the particle loss, which shows a good agreement with the calculation from Piwinski's formula. The beam distribution at the start and the end of simulation is shown in Fig. 2, in which the lost particles illustrate the aperture. The dependence of the 1-d quantum lifetime on the aperture is calculated by simulation and compared with Piwinski's formula, it shows a good agreement, as shown in Fig. 3.

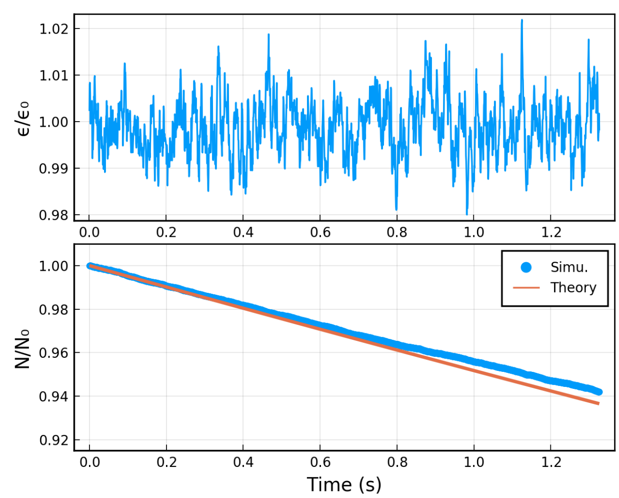

Figure 1: Evolution of beam emittance and particle number in the 1-d tracking ($A = 4.6$, $\lambda = 100\,s^{-1}$, $\tau_{theory} = 18.6\,s$).

Based on the same mothed, the 3-d particle tracking is also simulated, in which the ellipsoidal aperture (as defined by Eq. (5)) is used. The radiation damping rate is the same in all three dimensions with $\lambda_x = \lambda_y = \lambda_z = 100\,s^{-1}$. The results with different apertures are shown in Fig. 4, and they

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are consistent with the calculation from Eq. (10) (dashed line).

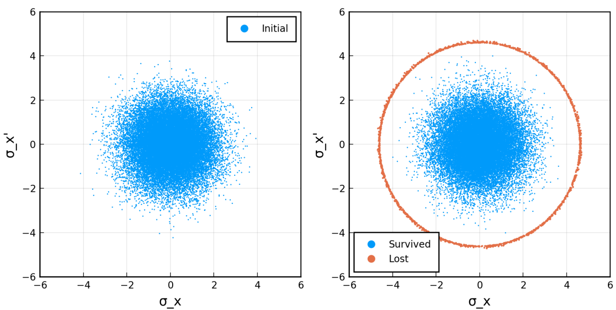

Figure 2: Beam distribution at the start (left) and the end (right) of simulation. The lost particles illustrate the aperture which is 4.6σ.

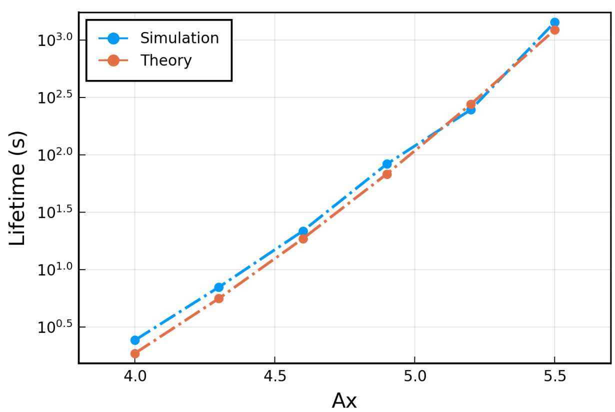

Figure 3: Comparison of the 1-d quantum lifetime between simulation and Piwinski's formula.

We see that for the aperture $A_x/A_y/A_z = 4.6/18/18$, the vertical and longitudinal aperture is large enough that the quantum lifetime is almost only determined by the horizontal aperture, which mean the simulation is close to the 1-d result. Comparing with Fig. 1, we see that the two results are very close as expected. It is worth noting that the coupling effect caused by dispersion is not included in the simulation, which results in the particle loss underestimated during tracking, especially for the simulation with small aperture as shown in Fig. 4. However, the 1-d and 3-d simulation results show good agreement with the theoretical formula, which

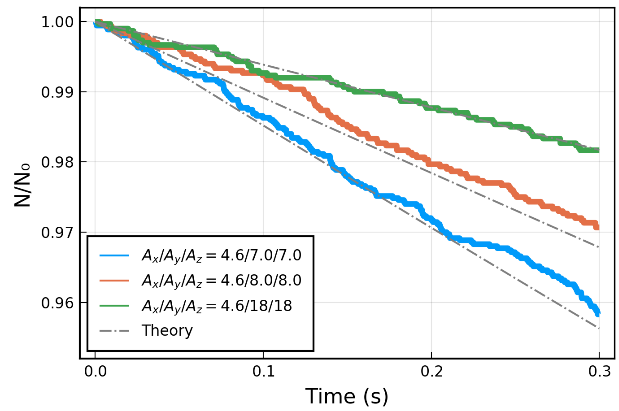

Figure 4: Simulation result of the particle loss with 3-d aperture, the dashed line is calculated by Eq.(10). The theoretical lifetime is 16.3 s, 9.2 s and 6.7 s, respectively ($\lambda_x = \lambda_y = \lambda_z = 100\ s^{-1}$).

demonstrate the 3-d quantum lifetime formula is correct and reasonable for the beam lifetime estimation.

SUMMARY

In this paper, we extended Piwinski's 1-d quantum lifetime formula to 3-d form, in which the ellipsoidal aperture is considered. The derivation of the 3-d quantum lifetime is detailed, and the formula shows a good agreement with the particle simulation. This 3-d lifetime formula can be more accurate for the electron machines with the limitation of physical aperture, dynamic aperture and momentum acceptance.

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