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ESTIMATES OF DAMPED EQUILIBRIUM ENERGY SPREAD AND EMITTANCE IN A DUAL ENERGY STORAGE RING*

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Abstract

A dual energy storage ring design consists of two loops at markedly different energies. As in a single-energy storage ring, the linear optics in the ring design may be used to determine the damped equilibrium emittance and energy spread. Because the individual radiation events in the two rings are different and independent, we can provide analytical estimates of the damping times in a dual energy storage ring. Using the damping times, the values of damped energy spread, and emittance can be determined for a range of parameters related to lattice design and rings energies. We present analytical calculations along with simulation results to estimate the values of damped energy spread and emittance in a dual energy storage ring. We note that the damping time tends to be dominated by the damping time of the high energy ring in cases where the energy of the high energy rings is significantly greater than that of the low energy ring.

INTRODUCTION

The dual energy storage ring design is a novel concept in the field of accelerator science with many possible applications. Our study focuses on the possible cooling application of dual energy storage ring where electron beams undergoing natural synchrotron radiation damping can be used to cool the ion beams [1]. The dual energy storage ring cooler design consists of a damping ring (high energy section) and a cooling ring (low energy section) at markedly different energies connected by an energy recovering superconducting RF structure as shown in Fig. 1.

We have already understood and established the stability criteria in a dual energy storage ring which is verified both analytically and using particle tracking simulation [2, 3]. A cooling application requires low emittance electron beams which can be achieved due to radiative cooling. Synchrotron radiation causes the emittances of the electron beam in all three degrees of freedom to damp towards equilibria. Thus, the equilibrium emittance and energy spread achieved in a dual energy storage ring are a balance between radiation damping and quantum excitation. We note that these equilibrium parameters tend to be dominated by the radiation in the high energy ring. In this paper, we provide a first estimate of the damped energy spread and emittance in a dual energy storage ring.

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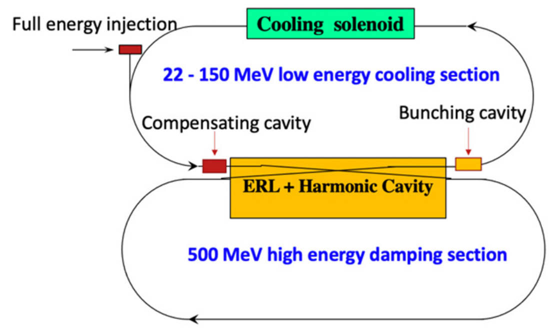

Figure 1: Schematic drawing of a dual-energy storage ring cooler.

DAMPED EQUILIBRIUM ENERGY SPREAD

Following the discussion in Wiedemann [4], the average change in synchrotron oscillation amplitude A given by a single photon emission event is

$$\langle \Delta A^2 \rangle = \langle A'^2 - A^2 \rangle = \varepsilon^2$$

where ε is the energy emitted by the photon and the averaging is over the synchrotron oscillation phase. If the emission rate for photons from a single electron is $\dot{N}_{ph}$, then a statistical argument gives that the growth rate in amplitude as

$$\frac{d\langle \Delta A^2 \rangle}{dt} = \dot{N}_{ph}\langle \varepsilon^2 \rangle \quad (1)$$

The average on the right-hand side of Eq. (1) is over the photon emission distribution and $\dot{N}_{ph}\langle \varepsilon^2 \rangle$ is defined by the following relation

$$\dot{N}_{ph}\langle \varepsilon^2 \rangle = \frac{55}{24\sqrt{3}} P_\gamma \varepsilon_c \quad (2)$$

Where P_γ is the synchrotron radiation power and it's average value is given by

$$\langle P_\gamma \rangle = \frac{U_0}{T_0} = \frac{c\, C_\gamma}{2\pi} \frac{E^4}{\rho^2}$$

where U_0 is the total energy radiated in synchrotron radiation and T_0 is the total revolution time in a storage ring. It is clear that the synchrotron radiation power depends on the beam energy E and the bend radius ρ. $C_\gamma = \frac{4\pi}{3} \frac{r_e}{(mc^2)^3} = 8.8463 \times 10^{-5} \frac{m}{GeV^3}$ and the critical photon energy is $\varepsilon_c = \hbar\omega_c = \frac{3\hbar c}{2(mc^2)^3} \frac{E^3}{\rho}$.

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In equilibrium, the oscillation damping balances this growth and

$$\langle A^2\rangle = \frac{\tau_z}{2}\,\dot{N}_{ph}\langle\varepsilon^2\rangle \rightarrow \sigma_E^2 = \frac{\tau_z}{4}\,\dot{N}_{ph}\langle\varepsilon^2\rangle,$$

because the square of the rms energy displacement of a particle undergoing sinusoidal energy motion is one half of the amplitude squared. The quantity τ_z is the damping time in the longitudinal dimension. Up to this point the argument applies equally to a dual energy storage ring. For the bend radiation, the integral over the emission distribution is possible analytically. In the case of a ring with bend radiation only, the damped energy spread is

$$\frac{\sigma_E^2}{E^2} = C_q\,\frac{\gamma^2}{2+\xi}\,\frac{\left\langle\frac{1}{\rho^3}\right\rangle}{\left\langle\frac{1}{\rho^2}\right\rangle} = \frac{\tau_z}{4E^2}\,\dot{N}_{ph}\langle\varepsilon^2\rangle \tag{3}$$

where

$$C_q = \frac{55}{32\sqrt{3}}\frac{\hbar c}{mc^2} = 3.84\ \times 10^{-13}\ m$$

Now, substituting the values of parameters in Eq. (3), the damped energy spread in a single energy ring case is given by

$$\frac{\sigma_E^2}{E^2} = \frac{\tau_z}{4E^2}\,\dot{N}_{ph}\langle\varepsilon^2\rangle = \frac{C_q}{\gamma^2}\,\frac{\left(\gamma^7\left\langle\frac{1}{\rho^3}\right\rangle\right)}{\left[\left((2+\xi)\gamma^3\left\langle\frac{1}{\rho^2}\right\rangle\right)\right]} \tag{4}$$

where, γ is the relativistic energy factor, ρ is the dipole bend radius and $(2+\xi)$ is the damping partition.

In the case of dual energy storage ring, one determines the equilibrium by adding total photon flux $\dot{N}_{ph}\langle\varepsilon^2\rangle$ from all sources in the two ring passes and utilizes the two-ring damping time τ_z determined elsewhere [5]. Therefore, the equilibrium energy spread is given by

$$\frac{\sigma_E^2}{E^2} = \frac{\tau_z}{4E^2}\dot{N}_{ph}\langle\varepsilon^2\rangle = \frac{C_q}{\hat{\gamma}^2}\,\frac{\left(\gamma_H^7\left\langle\frac{1}{\rho_H^3}\right\rangle+\gamma_L^7\left\langle\frac{1}{\rho_L^3}\right\rangle\right)}{\left[\left((2+\xi_H)\gamma_H^3\left\langle\frac{1}{\rho_H^2}\right\rangle+(2+\xi_L)\gamma_L^3\left\langle\frac{1}{\rho_L^2}\right\rangle\right)\right]} \tag{5}$$

where γ_L and γ_H are the corresponding relativistic energy factors for low energy ring and high energy ring, ρ_L and ρ_H are the bend radius for low energy ring and high energy ring, and $(2+\xi_L)$ and $(2+\xi_H)$ are the damping partition values for low energy ring and high energy ring respectively. In the denominator, $\hat{\gamma}^2$ is used which scales with the corresponding ring energy.

DAMPED EQUILIBRIUM EMITTANCE

Similar to the discussion leading to the equilibrium energy spread, we consider the perturbation to the transverse motion caused by photon emission. Since the photon emission will not change the particle position and direction [4]

$$\delta x = 0 = \delta x_\beta + D.\frac{u}{E} \rightarrow \delta x_\beta = -D.\frac{\varepsilon}{E}$$
$$\delta x' = 0 = \delta x'_\beta + D'.\frac{u}{E} \rightarrow \delta x'_\beta = -D'.\frac{\varepsilon}{E}$$

where D and D' are dispersion and the derivative of dispersion respectively. This phenomenon of photon emission will modify the phase-space ellipse and the variation of Courant – Snyder invariant is given by

$$\delta a^2 = \frac{2}{\beta_x}\left[Dx_\beta + \left(\beta_x D' - \frac{\beta_x'}{2}D\right)\left(\beta_x x' - \frac{\beta_x'}{2}\right)x\right]\frac{\varepsilon}{E} + \frac{1}{\beta_x}\left[D^2 + \left(\beta_x D' - \frac{\beta_x'}{2}D\right)^2\right]\left(\frac{\varepsilon}{E}\right)^2$$

The first term inside the bracket will vanish due to betatron oscillation. The average variation of the oscillation amplitude 'a' due to the emission of photon energy ε becomes

$$\delta\langle a^2\rangle = \mathcal{H}\,.\left(\frac{\varepsilon}{E}\right)^2$$

where $\mathcal{H}(z) = \beta D'^2 + 2\alpha DD' + \gamma D^2$ is the chromatic invariant of the ring.

To get the variation of the oscillation amplitude per turn, we average again over all photon energies, multiply by the total number of photons emitted per unit time and the integration is carried out over the whole ring

$$\Delta\langle a^2\rangle = \frac{1}{cE_0^2}\oint \dot{N}_{ph}\,\langle\varepsilon^2\rangle\mathcal{H}(z)dz.$$

and the rate of change of this oscillation amplitude is then

$$\frac{d\langle\Delta a^2\rangle}{dt} = \frac{1}{E_0^2}\,\left\langle\,\dot{N}_{ph}\langle\varepsilon^2\rangle\mathcal{H}(z)\right\rangle_z$$

where the index z indicates averaging along the ring. Since equilibrium energy spread is reached when the average quantum excitation rate around the ring is equal to the damping rate. In the same way, the equilibrium transverse emittance is reached when quantum excitation is equal to the damping. Equilibrium is reached when quantum excitation and damping are of equal strength, so

$$\frac{\sigma_u^2}{\beta_u} = \frac{\tau_u}{4E^2}\,\left\langle\,\dot{N}_{ph}\langle\varepsilon^2\rangle\mathcal{H}_u(z)\right\rangle_z$$

where $\sigma_u^2 = \langle u^2(z)\rangle = \left\langle\frac{1}{2}\,a^2\beta_u\right\rangle$ is the standard width of a Gaussian particle distribution with betatron function β_u. $u = x, y$ for both horizontal and vertical dimension. Now, the equilibrium beam emittance of a relativistic electron in a storage ring is given by

$$\epsilon_u = \frac{\sigma_u^2}{\beta_u} = C_q\,\frac{\gamma^2}{J_u}\,\frac{\left\langle\frac{\mathcal{H}_u}{\rho^3}\right\rangle}{\left\langle\frac{1}{\rho^2}\right\rangle} = \frac{\tau_u}{4E^2}\,\left\langle\,\dot{N}_{ph}\langle\varepsilon^2\rangle\mathcal{H}_u(z)\right\rangle_z \tag{6}$$

Substituting the values of τ_x, $\dot{N}_{ph}\langle\varepsilon^2\rangle$ and $\mathcal{H}_x(z)$ in Eq. (6), damped equilibrium emittance in a single energy ring becomes

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$$\epsilon_x = \frac{\tau_x}{4E^2}\left\langle \dot{N}_{ph}\langle\varepsilon^2\rangle\mathcal{H}_x(z)\right\rangle_z = \frac{C_q}{\hat{\gamma}^2}\frac{\left(\gamma^7\left\langle\frac{\mathcal{H}_x}{\rho^3}\right\rangle\right)}{\left[\left((1-\xi_x)\gamma^3\left\langle\frac{1}{\rho^2}\right\rangle\right)\right]} \quad (7)$$

where $(1-\xi_x)$ is the horizontal damping partition number.

However, $\delta\langle a^2\rangle$ is energy dependent (geometric emittance). As we know, in the dual energy storage ring, normalized emittance is a constant parameter, so we should get

$$\delta\langle\gamma a^2\rangle = \gamma\,\mathcal{H}\,.\left(\frac{\varepsilon}{E}\right)^2$$

Then,

$$\frac{d\langle\gamma a^2\rangle}{dt} = G_x^N = \frac{\gamma}{E^2}\oint \dot{N}_{ph}\langle\varepsilon^2\rangle\mathcal{H}(z)dz$$

$$\frac{d\langle\gamma a^2\rangle}{dt} = \frac{\langle \dot{N}_{ph}\langle\varepsilon_H^2\rangle\mathcal{H}_x^H\rangle}{m^2c^4\gamma_H} + \frac{\langle \dot{N}_{ph}\langle\varepsilon_L^2\rangle\mathcal{H}_x^L\rangle}{m^2c^4\gamma_L} \quad (8)$$

where $'L'$ and $'H'$ corresponds to the low energy ring and high energy ring respectively.

$$\therefore N\langle\varepsilon^2\rangle = \frac{3}{2}\,C_u\hbar c\frac{\gamma^3}{\rho^3}\frac{\langle P_\gamma\rangle}{\left\langle\frac{1}{\rho^2}\right\rangle}$$

where $C_u = \frac{55}{24\sqrt{3}}$. Substituting the value of average power loss due to synchrotron radiation, Eq. (8) takes the form

$$\therefore G_x^N = \frac{3}{4\pi T_{tot}}\,C_u\,C_\gamma\,\hbar\,m^2c^5\left(\gamma_H^6\left\langle\frac{\mathcal{H}_x^H}{\rho_H^3}\right\rangle + \gamma_L^6\left\langle\frac{\mathcal{H}_x^L}{\rho_L^3}\right\rangle\right) \quad (9)$$

Then equilibrium is reached with

$$\langle\gamma a^2\rangle = \frac{1}{2}\,\tau_x^t\,G_x^N, \sigma_x^2 = \frac{1}{2}\,\beta_x\langle a^2\rangle$$

The geometric emittance is given by

$$\epsilon_x = \frac{\sigma_x^2}{\beta_x} = \frac{1}{2}\frac{\langle\gamma a^2\rangle}{\gamma} = \frac{1}{4\gamma}\,\tau_x^t\,G_x^N$$

Substituting the values of τ_x^t and G_x^N in the above formula, the damped equilibrium emittance in a dual energy storage ring is given by

$$\epsilon_x = \frac{C_q}{\hat{\gamma}}\frac{\left(\gamma_H^6\left\langle\frac{\mathcal{H}_x^H}{\rho_H^3}\right\rangle+\gamma_L^6\left\langle\frac{\mathcal{H}_x^L}{\rho_L^3}\right\rangle\right)}{\left[\left((1-\xi_x^H)\gamma_H^3\left\langle\frac{1}{\rho_H^2}\right\rangle+(1-\xi_x^L)\gamma_L^3\left\langle\frac{1}{\rho_L^2}\right\rangle\right)\right]} \quad (10)$$

where $\mathcal{H}_x^L$ and $\mathcal{H}_x^H$ are chromatic invariants for low energy ring and high energy ring respectively. $(1-\xi_x^H)$ and $(1-\xi_x^L)$ are the horizontal damping partition numbers for high energy ring and low energy ring respectively.

RESULTS AND DISCUSSION

For the given low energy ring energy E_L and the high energy ring energy E_H, finally the following relationship is established

$$\left(\frac{\sigma_E}{E}\right)_L \approx \left(\frac{\sigma_E}{E}\right)_H \times \frac{\gamma_H}{\gamma_L} \quad (11)$$

where $\left(\frac{\sigma_E}{E}\right)_L$ and $\left(\frac{\sigma_E}{E}\right)_H$ are the damped equilibrium energy spread values for the low energy ring and high energy ring respectively. Same argument is applicable in the case of damped equilibrium emittance. Hence using Eq. (10), we get the following relationship

$$\epsilon_x^L \approx \epsilon_x^H \times \frac{\gamma_H}{\gamma_L} \quad (12)$$

where ϵ_x^L and ϵ_x^H are the corresponding horizontal damped equilibrium emittance values for low energy ring and high energy ring respectively. Note that for the low energy ring, it is determined by the anti-damping, other than equilibrium.

Damped equilibrium emittance and energy spread are obtained from the ELEGANT [6] particle tracking simulations and compared with the analytical calculations. Table 1 lists the results for the dual energy storage ring cooler system with the beam energies of 150 MeV in the cooling section (low energy ring) and 1000 MeV in the damping section (high energy ring). Overall, the results show a good agreement with 100 particles being used to extract the damped parameters. It is expected that the discrepancy can be further reduced with more particles statistically.

Table 1: Damped Equilibrium Emittance and Energy Spread from Analytical Calculations and Tracking Simulations

Parameter	Low Energy Ring		High Energy Ring	
	Analytical	Tracking	Analytical	Tracking
ϵ_x(μm)	18.20	19.99	2.73	2.55
Difference (%)	9.83		6.59	
$\frac{\sigma_E}{E}$ $(\times 10^{-3})$	3.03	3.28	0.454	0.509
Difference (%)	8.25		12.11	

CONCLUSION

We have estimated the damped equilibrium energy spread and emittance values in a dual energy storage ring cooler system with the beam energies of 150 MeV in the low energy ring and 1000 MeV in the high energy ring. The equilibrium emittance and energy spread values are in the absence of all the effects besides quantum excitation that would increase the emittance. It does not include space charge, longitudinal microwave instability, intra-beam scattering, etc. And their estimates are in the limit of low current. We use ELEGANT particle tracking code and do simulations for three damping times (1 million turns) with 500 particles initially. Damped equilibrium emittance and energy spread values from analytical calculations and tracking simulations are in good agreement.

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