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MAXIMUM ENTROPY RECONSTRUCTION OF 4D TRANSVERSE PHASE SPACE FROM 2D PROJECTIONS: WITH APPLICATION TO LASER WIRE MEASUREMENTS IN THE SNS HEBT*

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Abstract

We employ the principle of maximum entropy (MENT) to reconstruct 4D transverse phase space from its 2D projections. Emittance devices commonly measure two specific 2D projections, i.e. the horizontal and vertical phase space distributions. We show that: 1) given only these two 2D projections, their product is the analytic MENT solution to the 4D distribution; and 2) additional 2D projections provide information on inter-plane coupling in the MENT reconstruction of the 4D phase space which can be solved numerically.

At the Spallation Neutron Source (SNS), laser wires in the high energy beam transport (HEBT) enable non-invasive two-slit type transverse phase space measurements. Laser wires play the role of the first slit whereas physical wires downstream of a drift act as the second slit. We reconstruct the 4D phase space in the HEBT using all four horizontal/vertical permutations of the two "slits" where: 1) the two configurations with parallel slits constitute ordinary 2D phase space measurements in either plane; and 2) the two configurations with perpendicular slits carry coupling information.

INTRODUCTION

Tomography is the reconstruction of a multi-dimensional distribution from lower dimensional projections. The principle of maximum entropy (MENT) states that, among the many solutions for such an inverse problem, one should select the solution that maximizes the entropy of the resulting distribution.

MENT tomographic reconstruction of 2D and 4D transverse phase space distributions from 1D beam profile measurements was first formulated and applied at LANL [1, 2]. The technique has since been widely employed for phase space characterization. In this paper, we show that the same technique is readily extendable to the reconstruction of 4D transverse phase space distribution from 2D projections.

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MENT Tomographic Reconstruction

To reconstruct 4D phase space from 2D projections, the MENT approach seeks to maximize the entropy:

$$H[\rho] = -\iiiint \rho(x, x', y, y') \ln \rho(x, x', y, y') dx dx' dy dy', \tag{1}$$

given n constraint equations. The j-th constraint equation is given by:

$$\begin{aligned} G_j[\rho] &= g_j(u_j, u_j') - \iint \rho\left(\vec{x}\left(\vec{u}_j\right)\right) dv_j dv_j' \\ &= g_j(u_j, u_j') - \iint \rho\left(\mathbf{A}_j^{-1}\vec{u}_j\right) dv_j dv_j' \\ &= 0, \end{aligned} \tag{2}$$

where $g_j(u_j, u_j')$ is measured 2D projection. $\vec{u}_j$, the coordinates over which the j-th projection is taken, and $\vec{x}$, the transverse phase space coordinates of the beam at the reconstruction location, are assumed to be related by a linear transformation:

$$\vec{u}_j \equiv \begin{pmatrix} u_j \\ v_j \\ u_j' \\ v_j' \end{pmatrix} = \mathbf{A}_j \begin{pmatrix} x \\ y \\ x' \\ y' \end{pmatrix} \equiv \mathbf{A}_j \vec{x}. \tag{3}$$

The MENT solution can be found by introducing a new functional that contains Lagrange multiplier functions:

$$K[\rho] = H[\rho] + \sum_{j=1}^{n} \iint \lambda_j(u_j, u_j') G_j[\rho] dv_j dv_j'. \tag{4}$$

The variation of $K[\rho]$ by ρ should vanish, which leads to:

$$\rho = C_1 \exp\left(\sum_{j=1}^{n} \lambda_j(u_j, u_j') - 1\right) = C_2 \prod_{j=1}^{n} h_j(u_j, u_j'), \tag{5}$$

with C_1 and C_2 being constants. Hence the MENT solution is a product of n component functions h_j. The component functions can be found using the n constraint equations. Each constraint equation gives:

$$g_k(u_k, u_k') = h_k(u_k, u_k') \iint C_2 \prod_{j=1, j\neq k}^{n} h_j(u_j, u_j') dv_k dv_k'. \tag{6}$$

The component functions can be solved using a Gauss-Seidel type algorithm as shown in Eq. (7). The superscript denotes the iterative index for the component function, which shows that each updated function is immediately employed in all subsequent calculations.

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$$h_k^{(m+1)}(u_k,u_k') = \frac{g_k(u_k,u_k')}{\iint C_2 \prod_{j=1}^{k-1} h_j^{(m+1)}(u_j,u_j') \prod_{j=k+1}^{n} h_j^{(m)}(u_j,u_j') dv_k dv_k'}. \quad (7)$$

Two 2D projections of the 4D transverse phase space are routinely measured: the horizontal and vertical 2D phase space distributions. Given only these two 2D projections, MENT reconstruction has an analytical solution:

$$\rho(x,x',y,y') = \rho_{xx'}(x,x')\rho_{yy'}(y,y'). \quad (8)$$

This can be seen from Eq. (5) which states that, in this case, ρ has the form:

$$\rho(x,x',y,y') = f_1(x,x')f_2(y,y'). \quad (9)$$

A consequence of Eq. (8) is that any cross-plane 2D projection is the product of the respective 1D distributions, for example:

$$\rho_{xy}(x,y) = \iint \rho(x,x',y,y')dx'dy' = \rho_x(x)\rho_y(y). \quad (10)$$

APPLICATION TO SNS HEBT

The technique of 4D MENT tomography from 2D projections was applied to laser wire emittance measurements in the Spallation Neutron Source (SNS) high energy beam transport (HEBT) downstream of the superconducting linac. A schematic of the laser wire station is shown in Fig. 1. The station performs two-slit type emittance scans where the laser wires function as the 1st "slit" and physical wires downstream of a drift act as the 2nd "slit" to detect H^0 created by the laser [3, 4].

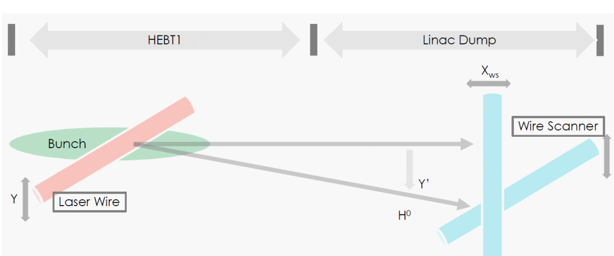

Figure 1: Schematic of the laser wire station in the SNS HEBT.

When the two slits are parallel, such a configuration performs the usual emittance scans whose results are the 2D phase space distributions:

$$\iint \rho\,(x_i,x_i',y_i,y_i')\,dy_i dy_i' = \rho_{x_i x_i'}(x_i,x_i') \quad (11)$$

$$\iint \rho\,(x_i,x_i',y_i,y_i')\,dx_i dx_i' = \rho_{y_i y_i'}(y_i,y_i'). \quad (12)$$

These measurements do not contain any information on the dependence between horizontal and vertical phase space coordinates, and the MENT solution would say there is no dependence as shown in Eq. (8). To obtain cross-plane information, two additional scans were made with perpendicular

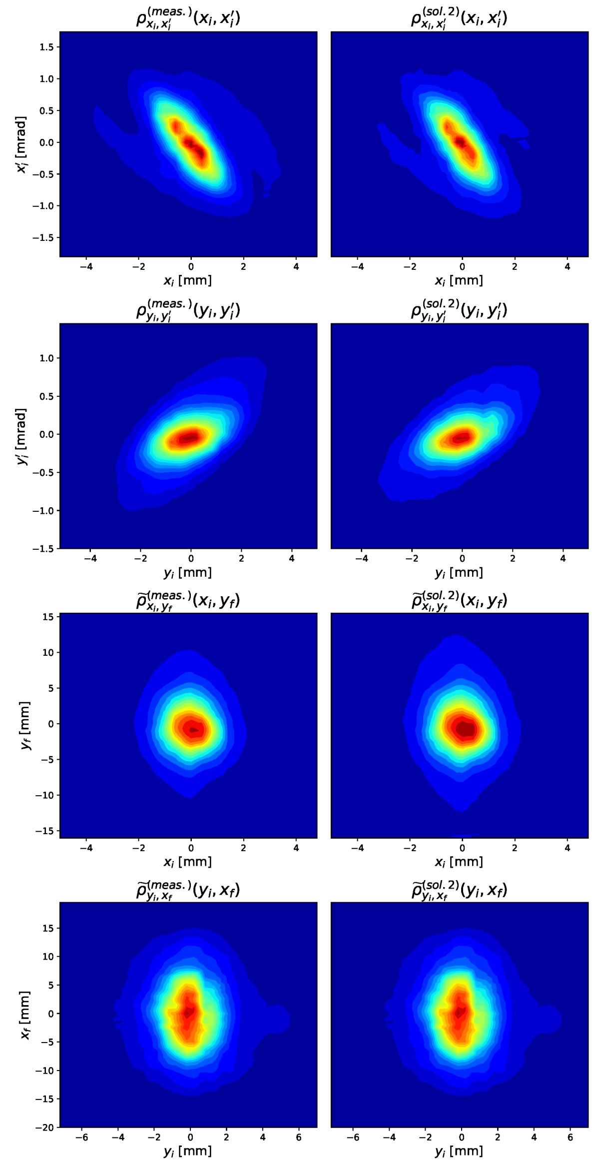

Figure 2: Four 2D scan results at the SNS HEBT laser wire station (left column, denoted by superscript "meas.") compared against the respective 2D projections of the 4D MENT solution (right column, denoted by superscript "sol. 2") obtained based on these scan results.

slits where the laser wire was horizontal but the physical wire was vertical, or vice versa. These measurements correspond to the following 2D projections:

$$\iint \rho\left(x_i, \frac{x_f - x_i}{L}, y_i, y_i'\right) dx_i dy_i' = \tilde\rho_{x_f y_i}(x_f,y_i) \quad (13)$$

$$\iint \rho\left(x_i, x_i', y_i, \frac{y_f - y_i}{L}\right) dx_i' dy_i = \tilde\rho_{x_i y_f}(x_i,y_f), \quad (14)$$

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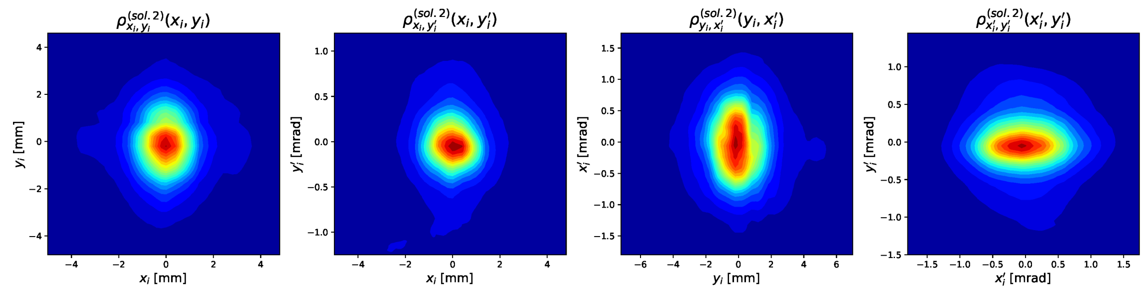

Figure 3: Cross-plane 2D phase space projections of the 4D MENT numerical solution obtained from four 2D projections at the SNS HEBT laser wire station.

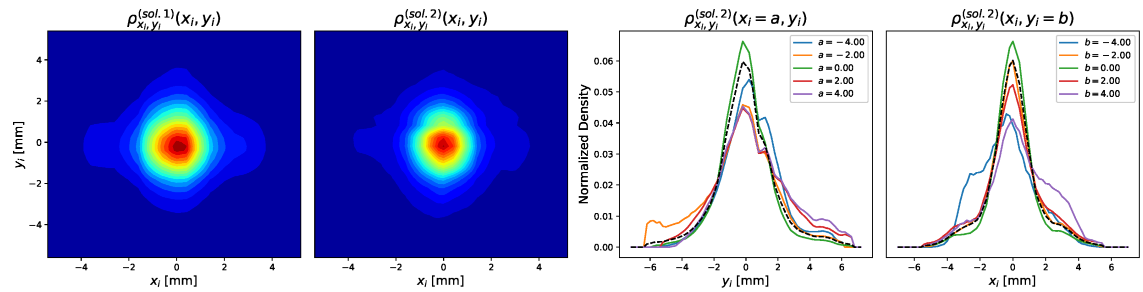

Figure 4: The superscript "sol. 1" denotes the two-projection (parallel slits only) MENT solution, and "sol. 2" denotes the four-projection (both parallel and perpendicular slits) MENT solution. The two plots on the left show the xy spatial profile of the beam as predicted by the respective 4D MENT solutions. The two plots on the right show the normalized density of 1D slices, taken at different values, of the four-projection solution. The black dotted line denotes the 1D projected density of the two-projection solution which is identical regardless of location (see Eq. (10)).

where the subscripts i and f denote the location of the laser wire and physical wire respectively. $L = 11.6$ m is the drift length between the two locations. Note that scan results with perpendicular slits are not distributions in phase space coordinates, so they are denoted by $\widetilde{\rho}$. All four 2D scan results are shown in Fig. 2.

Given the four 2D projections described by Eq. (11) to (14), the MENT solution to the 4D transverse phase space distribution was obtained numerically using the algorithm outlined in the previous section. The comparison in Fig. 2 shows that the numerical 4D MENT solution is valid because it correctly reproduces all four 2D measured projections.

The 4D MENT solution can be projected onto the four pairs of cross-plane coordinates whose 2D phase space distributions are not directly measured by the laser wire station. These results are shown in Fig. 3.

The additional information introduced by the two scans with perpendicular slits is illustrated by Fig. 4. In comparison against the two-projection MENT solution, the four-projection MENT solution has a more sophisticated structure as can be seen by how the 1D slice density varies with the value at which it is taken. In contrast, such a slice density is always the same for the two-projection MENT solution.

CONCLUSION

We discussed how to reconstruct 4D transverse phase space distribution from multiple 2D projections using the principle of maximum entropy. An analytical solution was presented for a two-projection special case and an algorithm was successfully implemented to obtain numerical MENT solution for general cases. We applied this tomography technique to the SNS HEBT laser wire station where, in addition to the two usual 2D scans with parallel slits, two 2D scans with perpendicular slits were employed to obtain cross-plane information. The four-projection MENT distribution was shown to have a more sophisticated structure than just a product of the horizontal and vertical 2D phase space distributions. How well this MENT solution agrees with the true phase space distribution will be a subject of further investigation. Once implemented, this technique has the potential to enable noninvasive online 4D tomography of an operational 1 GeV beam at the SNS which may improve understanding of the beam dynamics and enhance control of the beam at the stripping foil and target.

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REFERENCES

[1] G. N. Minerbo, O. R. Sander, and R. A. Jameson, "Four-dimensional beam tomography," *IEEE Transactions on Nuclear Science*, vol. 28, no. 3, pp. 2231–2233, 1981, doi:10.1109/TNS.1981.4331646

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[2] C. T. Mottershead, "Maximum entropy beam diagnostic tomography," *IEEE Transactions on Nuclear Science*, vol. 32, no. 5, pp. 1970–1972, 1985, doi:10.1109/TNS.1985.4333784

[3] Y. Liu *et al.*, "Laser wire beam profile monitor in the spallation neutron source (sns) superconducting linac," *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 612, no. 2, pp. 241–253, 2010, doi:10.1016/j.nima.2009.10.061

[4] Y. Liu *et al.*, "Nonintrusive emittance measurement of 1gev h– beam," *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol. 675, pp. 97–102, 2012, doi:10.1016/j.nima.2012.02.009