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COLLECTIVE (IN)STABILITY NEAR THE COUPLING RESONANCE*

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Abstract

We show how to treat transverse collective instabilities when operating in the vicinity of the coupling (or tune difference) resonance. We begin by defining the approximate independent degrees of freedom including both linear coupling and chromatic effects. We then show how the destabilizing force due to wakefields and the stabilizing chromatic effects can be described by a linear combination of the horizontal and vertical motion that depends upon how close one is to the resonance. The theory agrees well with tracking studies, and will be relevant for those next-generation storage rings that plan to operate near the coupling resonance to produce nearly round beams, including the multi-bend achromat upgrade for the Advanced Photon Source.

INTRODUCTION

Storage ring design usually begins with a lattice whose linear dynamics in the horizontal and vertical planes are independent (uncoupled). Skew quadrupoles are then added to set the coupling between the two directions to a desired (typically small) level. Even when the *x*-*y* coupling is small, however, it is well-known that the motion in the two planes becomes essentially inseparable if the vertical and horizontal tunes differ by approximately an integer. Motion near the linear coupling (or tune difference) resonance is stable with an equilibrium that has equal emittances in each plane. Several next-generation, low-emittance storage rings are planning to operate on the coupling resonance to produce nearly round particle beams with a longer Touschek lifetime.

The horizontal and vertical emittances are equal on the coupling resonance because the independent degrees of freedom are composed of equal parts motion in *x* and *y*. Similarly, other properties that are usually characterized by their horizontal and vertical components will become "mixed" near the coupling resonance. For example, the effects of horizontal and vertical wakefields will no longer be independent when $\nu_x \approx \nu_y$, and also the stabilizing role of chromaticity will depend upon its value along both *x* and *y*. This paper describes collective effects near the coupling resonance, showing that the collective dynamics can be characterized by wakefields and chromaticities that result from "sharing" those of the horizontal and vertical directions.

SKETCH OF THE THEORY

The first step of the theory is to identify the three independent degrees of freedom for single particle motion including transverse coupling and chromatic effects. Our first step along this path prepares to decouple the *x*-*y* motion in the usual way by defining the complex coordinates

$$u_x(s) = \sqrt{\mathcal{J}_x}\, e^{i\Psi_x} e^{-2i\pi(\nu_x - \Delta_\nu/2)s/\mathscr{C}_R} e^{i\varphi_r/2} \tag{1}$$
$$u_y(s) = \sqrt{\mathcal{J}_y}\, e^{i\Psi_y} e^{-2i\pi(\nu_y + \Delta_\nu/2)s/\mathscr{C}_R} e^{-i\varphi_r/2}, \tag{2}$$

where $\mathcal{J}_{x,y}$ are the horizontal (vertical) actions in a smooth focusing lattice, $\Psi_{x,y}$ are the angles, $\mathscr{C}_R$ is the ring circumference, and $\Delta_\nu = \nu_x - \nu_y$ is the tune difference; the phase φ_r is associated with the phase of the skew quad settings such that within the single resonance approximation the skew quads result in the coupling $\kappa u_x u_y^*/2 + c.c.$, where κ is proportional to the skew quadrupole strength. In the transverse plane the single particle equations of motion are

$$u_x' - 2\pi i(\Delta_\nu/2 - \xi_x p_z)u_x/\mathscr{C}_R = (i\kappa/2\mathscr{C}_R)u_y \tag{3}$$
$$u_y' + 2\pi i(\Delta_\nu/2 - \xi_x p_z)u_y/\mathscr{C}_R = (i\kappa/2\mathscr{C}_R)u_x. \tag{4}$$

We want a new set of three independent degrees of freedom that eliminates the *x*-*y* coupling $\propto \kappa$ and the chromatic dependence $\propto \xi_{x,y}p_z$. For example, in the absence of chromaticity these are linear combinations of u_x and u_y (see, e.g., [1–3]) that we can write as $u_+ = u_x \cos\theta + u_y \sin\theta$ and $u_- = -u_x \sin\theta + u_y \cos\theta$, where the coupling angle θ is defined by $\tan 2\theta = \kappa/(2\pi\Delta_\nu)$ with $0 \le \theta \le \pi/2$.

On the other hand, when $\kappa \to 0$ we can remove the chromatic dependence by introducing the head-tail (chromatic) coordinates $\tilde{u}_x = u_x e^{ik_{\xi x}z}$ and $\tilde{u}_y = u_y e^{ik_{\xi y}z}$. These differ from the $\xi = 0$ coordinates by the head-tail phases [4, 5] that are proportional to the chromaticity and given by

$$k_{\xi x}z = (2\pi\xi_x/\alpha_c\mathscr{C}_R)z \quad k_{\xi y}z = (2\pi\xi_y/\alpha_c\mathscr{C}_R)z. \tag{5}$$

In the general case we can approximately decouple the degree of freedom by defining $k_+ = k_{\xi x}\cos^2\theta + k_{\xi y}\sin^2\theta$ and $k_- = k_{\xi x}\sin^2\theta + k_{\xi y}\cos^2\theta$ and introducing the coordinates

$$\begin{bmatrix} u_+ \\ u_- \end{bmatrix} = \begin{bmatrix} e^{ik_+z}\cos\theta & e^{ik_+z}\sin\theta \\ -e^{ik_-z}\sin\theta & e^{ik_-z}\cos\theta \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}. \tag{6}$$

The new coordinates (u_+, u_-) match the usual set far from the coupling resonance ($\theta \to 0, \pi/2$) or when $|k_{\xi x}z|, |k_{\xi y}z| \ll 1$; any coupling between the two is negligibly small provided the chromatic tune difference is much less than coupled oscillation tune, $\sigma_\delta \left|\xi_x - \xi_y\right| \ll \kappa$.

Including wakefields in the $u_\pm$ dynamics leads to

$$u_+' = +i\omega_\kappa u_+ + i(\partial V_{\text{wake}}/\partial u_+^*) \qquad z' = \alpha_c p_z \tag{7}$$
$$u_-' = -i\omega_\kappa u_- + i(\partial V_{\text{wake}}/\partial u_-^*) \qquad p_z' = -F_z, \tag{8}$$

where the coupled frequency $\omega_\kappa = \sqrt{\kappa^2 + (2\pi\Delta_\nu)^2}/2\mathscr{C}_R$, F_z is the longitudinal force, and V_{wake} is a rather complicated

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function. It turns out that if we define the "coupled mode" wakefield

$$W_+(z) = \cos^2\theta\, W_x(z) + \sin^2\theta\, W_y(z), \tag{9}$$

then one can show that the wakefield force

$$\begin{aligned}\frac{\partial V_{\text{wake}}}{\partial u_+^*} \propto u_+ \int d\hat{z} d\hat{p}_z\, e^{ik_+(z-\hat{z})} W_+(z-\hat{z}),\\ - u_- \int d\hat{z} d\hat{p}_z\, e^{i(k_+ z - k_- \hat{z})} W_c(z-\hat{z}),\end{aligned} \tag{10}$$

where $W_c(z) = \sin(2\theta)\left[W_y(z) - W_x(z)\right]$ can typically be neglected. This shows that the stability of u_+ is governed by the effective chromaticity ξ_+ and wakefield W_+. The mode u_- is similar, but in this case the head-tail phase $k_- z$ comes into play with the wakefield

$$W_-(z) = \sin^2\theta\, W_x(z) + \cos^2\theta\, W_y(z). \tag{11}$$

We could take this further to derive a set of stability equations like those solved with modal decomposition [6,7]. However, here we proceed to illustrate the essential dynamics using `elegant` [8] tracking simulations.

EFFECTIVE CHROMATICITY

Our first examples will illustrate the role of differing chromaticities on collective effects when $\nu_x \approx \nu_y$. Table 1 shows that the APS-U lattice has rather large and different chromaticities in the horizontal and vertical planes, which were the result of a genetic optimization procedure designed to maximize the x-ray brightness, Touschek lifetime, and dynamic aperture including errors. The final values of $\xi_x \approx 8.1$ and $\xi_y \approx 4.7$ imply that collective stability will depend upon how close the tunes are to the coupling resonance.

Table 1: Parameters for the APS-U Storage Ring

Parameters	Values
Tunes (ν_x, ν_y)	(95.1, 36.1)
Damping times (τ_x, τ_y, τ_z)	(10.3, 23.1, 30.8) ms
Natural emittance ε_0	41.66 pm
Dimensionless coupling κ	0.164
Chromaticities (ξ_x, ξ_y)	(8.106, 4.723)
Momentum compaction α_c	4.04×10^{-5}
RMS energy spread σ_δ	0.135%
RMS length in double rf	53.6 ps

In particular, the effective chromaticity governing stability will be a linear combination of ξ_x and ξ_y that is equal to the smaller of ξ_+ and ξ_-. Far from the tune resonance the beam is unstable in the vertical plane, and since $0 \le \theta \le \pi/2$ with $\theta = \pi/4$ on the coupling resonance we find that

$$\xi_{\text{eff}} = \tfrac{1}{2}\big(1 + |\cos(2\theta)|\big)\xi_y + \tfrac{1}{2}\big(1 - |\cos(2\theta)|\big)\xi_x \tag{12}$$

for the APS-U. The effective chromaticity is ξ_y far from the resonance, and increases the tunes approach each other to a maximum of $\xi_{\text{eff}} = (\xi_x + \xi_y)/2$ when $\nu_x = \nu_y$. Hence, the instability threshold current also increases near the tune resonance, which we illustrate in this section by setting the wakefields to be equal in the two planes.

We show results of APS-U tracking simulations including the resistive wall impedance in Fig. 1. The red line labelled "Coupled tracking" plots the instability threshold current obtained from the coupled tracking. Far from the tune resonance the threshold $I_{\text{thresh}} \approx 10.5$ mA, but it increases as the fractional tunes approach each other to approximately 13.7 mA near $\Delta_\nu = 0$. In addition, the blue line plots the instability current obtained from `elegant` tracking in an uncoupled lattice whose chromaticity equals the ξ_{eff} from Eq. (12). For example, at $\Delta_\nu = \pm 0.01$ we have $\cos(2\theta) \approx \pm 0.358$, and uncoupled `elegant` tracking using $\xi_y = \xi_{\text{eff}} \approx 5.81$ predicted $I_{\text{thresh}} \approx 12.6$ mA. The results from uncoupled tracking with ξ_{eff} in blue agrees quite well with the predictions from the coupled tracking in red, and we see that the instability threshold current changes by ~ 30% because of the disparate chromaticitities.

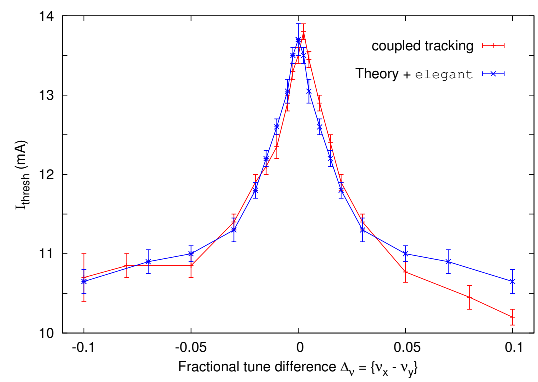

Figure 1: Effect of unequal chromaticities on collective stability near the coupling resonance. Red lines plot the threshold current obtained from the coupled model in `elegant`, while the blue lines plot the I_{thresh} for an uncoupled lattice whose chromaticity is given by the effective value Eq. (12).

EFFECTIVE WAKEFIELD

In this section we investigate how the coupling resonance leads to a "sharing" of the wakefields in a manner very similar to that of the chromaticity, namely that of Eqs. (9)-(11). If stability in the uncoupled system is dictated by the vertical wakefield W_y, then near the tune resonance the unstable dipole motion is driven by the effective wakefield

$$W_{\text{eff}} = \big(1 + |\cos(2\theta)|\big)\frac{W_y}{2} + \big(1 - |\cos(2\theta)|\big)\frac{W_x}{2}. \tag{13}$$

Again, when the $\nu_x = \nu_y$ the effective wakefield will be the arithmetic mean of the horizontal and vertical wakefields.

We ran additional simulations of the APS-U to investigate the role of the effective wakefield W_{eff}. Here we again used the parameters of Table 1, but set $\xi_x = \xi_y$ to isolate the shared wakefield effect from that of the chromaticity. In addition, we apply the same resistive wall impedance in the

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vertical plane as before, but now reduce W_x to see how it changes stability near the coupling resonance.

We show simulation results in Fig. 2, where the red plots `elegant` predictions when $W_x \to 0$, while the blue assumes that the horizontal wakefield is half of that in the vertical. The former is a somewhat artificial example for illustration, while the latter applies to a ring that has flat chambers and similar beta functions in each plane. Black lines plot the theory assuming that the uncoupled $I_{\text{thresh}} = 10.6$ mA and that the effective wakefield is given by (13). The simulation results show a clear increase of the threshold current near the tune resonance as wakefield sharing becomes important. The tracking agrees quite well with the theory when $W_x = W_y/2$, although the maximum is shifted slightly to negative detunings. When $W_x = 0$ this shift to $\Delta_\nu < 0$ is even more pronounced, and we believe that it can be explained by the additional coupling between the two modes that is provided by the coupling wake $W_c \propto W_y - W_x$. Regardless, it is clear that the coupled modes respond to an effective W_{eff} that depends upon both the vertical and horizontal wakefields as given by Eq. (2).

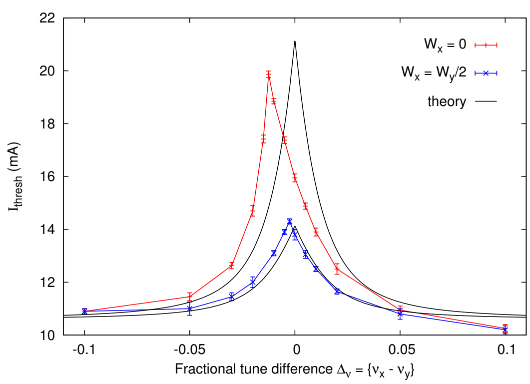

Figure 2: Role of both wakefields when $\Delta_\nu \approx 0$. The simulations use the same parameters and W_y as Fig. 1, but set $\xi_x = \xi_y = 4.723$. Coupled tracking in `elegant` gives the stability thresholds plotted by the red and blue lines when assuming that the horizontal wakefield $W_x = 0$ and $W_x = W_y/2$, respectively. The black theory lines plots the prediction for the effective or "shared" wakefield from Eq. (13) using the observed, uncoupled $I_{\text{thresh}} = 10.6$ mA.

COUPLED DYNAMICS FOR THE APS-U

Our final example includes the impact of both the chromatic and wakefield "sharing" near the coupling resonance. Specifically, we will use the full transverse impedance for the APS-U including both the geometric and resistive wall components. In addition, we add the longitudinal impedance with a `ZLONGIT` element, and more faithfully model the rf systems using `RFMODE` elements and the `BUNCHED_BEAM_MODE` feature to model the induced voltage from the other 47 equally-spaced "pseudobunches". Finally, we will keep the resulting longitudinal potential constant for all simulations by (somewhat artificially) fixing the current seen by the `RFMODE` and `ZLONGIT` elements at a constant value even as we increase the current for `ZTRANSVERSE`. The final result of our longitudinal modeling is a bunch that is "overstretched" to an rms length of 100 ps, and whose energy spread is somewhat increased by the microwave instability to $\sigma_\delta \approx 0.15\%$.

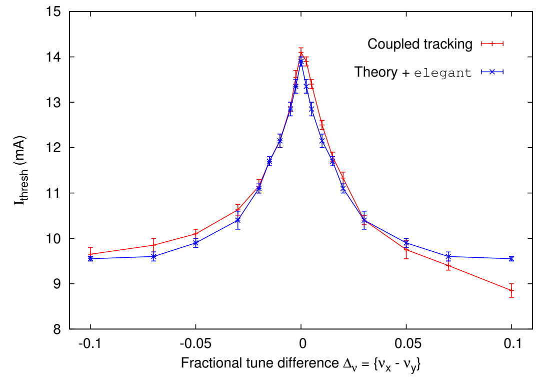

Figure 3: Transverse instability threshold current as a function of fractional tune difference for the full (geometric plus resistive wall) APS-U transverse impedance.

The red line in Fig. 3 plots the simulated instability threshold current as a function of fractional tune difference. The simulations agree quite closely with the blue line derived from `elegant` results obtained using an uncoupled lattice with the effective chromaticity ξ_{eff} from Eq. (12) and the effective wakefield W_{eff}^β from (13). The difference between the maximum stable current at $\Delta_\nu \approx 0$ and the uncoupled I_{thresh} is somewhat larger than that of Fig. 1 because the latter only benefits from the fact that $\xi_x > \xi_y$, while the increase of I_{thresh} in Fig. 3 also comes because $W_x^\beta < W_y^\beta$. In addition, we see that the uncoupled threshold current from the full APS-U impedance in Fig. 3 is only ~1 mA less than the purely resistive wall wakefield of Fig. 1. This is because the reduction in stability due to the larger wakefield is partially compensated by the lower peak current and longer bunch length provided by the longitudinal impedance and the overstretched rf potential. Using the full wakefield with the flattened potential of Fig. 1 reduces the uncoupled I_{thresh} to about 6.6 mA.

CONCLUSIONS

We have obtained approximately independent degrees of freedom for a weakly-coupled lattice operating near the coupling resonance, and used them to show that collective stability is governed by an effective wakefield Eq. (13) and chromaticity Eq. (12), each of which is a linear combination of the uncoupled values in the horizontal and vertical planes. Our theoretical predictions agree quite well with tracking simulations in `elegant`.

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